POLITECNICO MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE

A variational approach to Pontryagin's Maximum Principle for stochastic optimal control and stopping problems

MSc in Mathematical Engineering

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Academic Year 2024–2025

Abstract

This thesis develops a variational framework for deriving Pontryagin's Maximum Principle in stochastic optimal control and optimal stopping problems. We establish necessary optimality conditions for controlled stochastic differential equations using functional-analytic methods, specifically exploiting the differentiability of the control-to-state map and adjoint calculus via backward stochastic differential equations. The main contributions include: (i) a rigorous proof of the stochastic PMP using Gâteaux and Fréchet differentiability; (ii) an extension to problems with random terminal times, where the stopping time itself becomes part of the control; and (iii) applications to quadratic control problems and natural resource valuation. Even if the Dynamic Programming approach provides a global value-function characterization and, under regularity, sufficiency via the HJB equation, it often faces the curse of dimensionality and relies on viscosity-type solutions which may be hard to deal with.

Keywords: Pontryagin Maximum Principle; Stochastic Optimal Control; Variational Methods; Backward Stochastic Differential Equations; Necessary Conditions for Optimality; Random Terminal Time; Hamiltonian Systems

Abstract

Questa tesi sviluppa un quadro variazionale per derivare il Principio del Massimo di Pontryagin (PMP) nel contesto del controllo ottimale e dell'arresto ottimale stocastico. Saranno stabilite le condizioni necessarie di ottimalità per le equazioni stocastiche differenziali controllate utilizzando metodi funzionali e analitici, sfruttando, in particolare, la differenziabilità della mappa controllo-stato e il calcolo aggiunto attraverso delle equazioni stocastiche differenziali a ritroso. I principali contributi includono: (i) una dimostrazione rigorosa del PMP stocastico utilizzando la differenziabilità di Gâteaux e Fréchet; (ii) un'estensione ai problemi nei quali il tempo finale è casuale e fa parte del controllo stesso; e (iii) applicazioni a problemi di controllo quadratico e valutazione delle risorse naturali. Anche se l'approccio della Programmazione Dinamica fornisce una caratterizzazione globale della funzione di valore e, in condizioni di sufficiente regolarità, le condizioni sufficienti tramite la soluzione delle equazioni HJB, spesso si trova ad affrontare il problema della dimensionalità; inoltre, si basa su tecniche che comprendono le soluzioni di viscosità che possono risultare difficili da trattare.

Contents

Introduction

Uncertainty is deeply rooted in most real-world systems. It puts a fundamental barrier to human beings' ability to make predictions, plan and optimize the result of dynamic processes. The thesis mainly deals with dynamic systems, i.e., that evolve continuously over time according to laws influenced by both deterministic mechanisms and random perturbations. These systems are represented by Itô's stochastic differential equations (SDEs), in which white noise, a representation of totally independent random forces acting on the system, is the principal source of randomness. Since these systems are dynamic, the relevant decisions (controls), which are made based on the most updated information available to the decision makers, must also change over time (Yong & Zhou, 1999).

The inception of optimal control theory can be traced back to the work of the Russian mathematician Lev Pontryagin and his collaborators in the 1950s (Pontryagin et al., 1962), which was named *Pontryagin's Maximum Principle* (PMP). It was the first significant step toward an analytic framework for deterministic systems. It allowed the transformation of an optimization problem in function spaces into the study of a system of state and adjoint differential equations linked through a Hamiltonian. Its structure resembled the Euler-Lagrange equations in calculus of variations but it generalized them to the cases where control could be linear or nonlinear.

Stochastic optimal control came about as the next step after deterministic systems have incorporated randomness. Kushner (1967), Bismut (1973) and Haussmann and Pardoux (1986) developed the first stochastic analogues of PMP by adapting Pontryagin's framework to Itô diffusion processes. The choice of introducing backward stochastic differential equations (BSDEs) made by Pardoux and Peng (1990) was key in providing the representation of the adjoint process in stochastic control problems and, later, Yong and Zhou (1999) completed the theory formalization. Since then, the PMP framework has been extended to various settings.

A central methodological distinction in optimal control theory lies between the Pontryagin Maximum Principle and the *Dynamic Programming* (DP) approach. Both approaches deal with the same class of optimization problems but differ fundamentally in their mathematical formulation and the kind of results they produce.

The point of view of Pontryagin Maximum Principle is to provide the necessary conditions for optimality, by means of a variational approach. An infinitesimal change of the control is analyzed in a variational manner: the main outcome of this analysis is the set of the first-order optimality conditions which characterize optimal control. The result has the form of a system of forward–backward differential equations, the state equation and the adjoint equation which are linked through the Hamiltonian minimized pointwise with respect to the control. The local variational structure of the problem is particularly well-suited to analytic and functional-analytic methods.

On the other hand, the DP method, introduced by Bellman (1958), operates on the principle of recursion. It bases its arguments on the value function that denotes the minimal expected cost achievable from a given state and time and the Hamilton-Jacobi-Bellman (HJB) partial differential equation which the value function satisfies. The HJB equation can serve both as a complete set of necessary and sufficient conditions for optimality in case the value function is sufficiently smooth. Nevertheless, in case of stochastic or high-dimensional systems, the HJB equation can generally be very difficult to solve, both analytically or numerically, and the value function may lack the regularity required even very simple cases. To deal with this difficulty, Crandall and Lions (1983) introduced the notion of viscosity solutions for first order equations but requires discretizations of the state space, which scales poorly with dimension.

These two methodologies differ in several aspects, but they are still very much related: under suitable smoothness assumptions, the gradient of the value function in the dynamic programming setting is the same as the adjoint variable in the Pontryagin setting, and both reflect the same Hamiltonian structure. However, a variational method provides a local and differential description of optimality, whereas the dynamic programming method provides a global and recursive description via the value function.

In this thesis, we use the Pontryagin Maximum Principle rather than the Dynamic Programming for several reasons. First, the PMP fits naturally with the variational and functional-analytic methodology developed in this thesis: by showing the optimality through differentiability and duality arguments in Banach and Hilbert spaces, which form the mathematical foundation of this work. Secondly, the PMP eliminates the requirement to solve the HJB equation, the analytical and numerical challenge of which escalates rapidly with the state space dimensions. Thirdly, the PMP formulation leads to a clear and flexible framework of state, adjoint, and Hamiltonian, which can be extended to a variety of nonstandard scenarios, for instance, systems with random terminal times, fractional dynamics, and non-convex control constraints. Therefore, the PMP serves as a more natural and comprehensive background for the theoretical and variational advances presented in this thesis.

Recent research on the PMP reveals a dynamical expansion of its theoretical foundations and practical applications. The development of discrete-time versions and extensions to rate-constrained and non-smooth optimal control problems has broadened its relevance to modern engineering systems: Ganguly et al. (2025)Wachsmuth (2024). In quantum control, PMP offers robust solutions for time-optimal maneuvers under uncertainty, Fresse-Colson et al. (2025), while mean-field formulations are advancing machine learning theory by linking control concepts with training dynamics: Kamtue et al. (2024), E et al. (2018).

As far as novelty is concerned, the research here does not merely restate existing results but it positions itself at being a self-contained framework that applies PMP directly to optimal stopping, considering the final time as part of control and deriving first-order necessary conditions for its optimality. Moreover, it preserves the classical Hamiltonian structure, including an adjoint equation. While Yang (2022) and Yang (2020) researched particular cases of random terminal time settings, no prior work has provided a general PMP formulation that covers the full class of random terminal times.

Let us now sketch the main concepts of each chapter of the book.

Chapter 1: Since the thesis is supposed to be self-contained, it consists of some prerequisite topics in functional analysis and stochastic calculus. Specifically, this chapter is a collection of notions and results that are scattered around in the literature and are related to stochastic controls. It also unifies terminology and notation, which are going to be used in the following chapters.

Chapter 2: This chapter presents BSDEs. This topic was not covered during any course and needed to be introduced extensively. Although BSDEs extend beyond the immediate scope of stochastic control, they provide the essential analytical foundation for the adjoint equations developed in subsequent chapters. It begins with the argument of why we can't treat BSDEs in the same way as we treat SDEs. After that, the existence and uniqueness of the solutions to nonlinear BSDEs is investigated for two types of time durations, finite deterministic horizon and random horizon.

Chapter 3: In this chapter the stochastic optimal control problem without stopping time is formulated. We are concerned with the control of the state equation SDE

$$\mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \sigma(t, X_t, u_t)\mathrm{d}W_t, \qquad X_0 = x_0 \in \mathbb{R}^n.$$

We want to optimize with respect to the cost functional

$$J(u, X) := \mathbb{E}\left[\int_0^T f(s, X_s, u_s)\mathrm{d}s + h(X_T)\right],$$

where the controls u take values in a set of admissible controls with some regularities. Under some regularities assumptions on the coefficients, we will assess the differentiability of the cost functional with respect to the control. To do so we need to introduce an adjoint BSDE, with $\theta_t = (X_t, u_t)$

$$\begin{cases} \mathrm{d}p_t = \left[-p_t^T D_x b(t, \theta_t^u) - \mathrm{Tr}(q_t^T D_x\sigma(t, \theta_t^u)) - D_x f(t, \theta_t^u)\right]\mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = D_x h(X_T^u). \end{cases}$$

A Hamiltonian is then introduced to represent the gradient in terms of the adjoint equation:

$$H(t, x, u, p, q) = p^T b(t, x, u) + \mathrm{Tr}(q^T \sigma(t, x, u)) + f(t, x, u),$$

hence we will obtain the following representation:

$$D\widetilde{J}(u) = \mathbb{E}\left[\int_0^T D_u H(t, X_t^u, u_t, p_t, q_t)\mathrm{d}t\right],$$

where $\widetilde{J}(u) := J(u, X^u)$ and X^u is the solution to the state equation associated with the control u. At last we will state and prove Pontryagin's Maximum Principle.

Chapter 4: An extension of the PMP of the situation where the terminal time being part of the control is presented here. The discussion is split into two parts, the former being concentrated on the running part of the cost and the latter on the terminal cost. We are concerned with a state equation which evolves until a stopping time τ with respect to some filtration:

$$\mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \sigma(t, X_t, u_t)\mathrm{d}W_t, \qquad X_0 = x_0 \in \mathbb{R}^n.$$

We want to optimize with respect to the cost functional

$$J(u, \tau, X) := \mathbb{E}\left[\int_0^\tau f(s, \tau, X_s, u_s)\mathrm{d}s + h(X_\tau)\right],$$

where the control is the pair (u, τ), taking values in suitable sets of admissible controls with some regularities. Again, under some regularities assumptions on the coefficients, we will assess the differentiability of the cost functional with respect to the control. We introduce an adjoint BSDE with random final condition, with $\theta_t = (X_t, u_t)$

$$\begin{cases} dp_t = -\Big(D_x f(t, \tau, \theta_t) + D_x b(t, \theta_t)^T p_t + \mathrm{Tr}(q_t^T D_x \sigma(t, \theta_t))\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_\tau = D_x h(X_\tau), \end{cases}$$

Finally, an extended Hamiltonian is introduced to represent the gradient of the cost functional:

$$\mathcal{H}(t, \tau, x, u, p, q) := \mathbb{1}_{\{t \le \tau\}} f(t, \tau, x, u) + \mathbb{1}_{\{t \le \tau\}} \langle b(t, x, u), p\rangle + \mathrm{Tr}(q^\top \sigma(t, x, u)),$$

hence we will obtain:

$$\begin{aligned} D\widetilde{J}(u, \tau)[v, \rho] = \mathbb{E}\Bigg[&\int_0^T D_u \mathcal{H}(\theta_t, p_t, q_t)[v] + D_\tau \mathcal{H}(\theta_t, p_t, q_t) \cdot \rho\,\mathrm{d}t \\ &+ \frac{1}{2}\mathrm{Tr}\left(\sigma(\tau, \theta_\tau)^T h_{xx}(X_\tau)\sigma(\tau, \theta_\tau)\right)\rho\Bigg]. \end{aligned}$$

And we will state and prove the Extended PMP under this new setting.

Chapter 5: In this chapter, we show that our results are applicable to the quadratic case, a general terminal time control case as well as the pricing of natural resources. In the latter instance, we verify that the solutions given by the Dynamic Programming satisfy the necessary conditions derived in the thesis.

Future developments of this thesis are many. A first direction is to establish existence of optimal controls theorems for stochastic control problems, jointly controlling the control and the stopping time. This may need structural assumptions such as compactness/closeness of the admissible control set, lower semi-continuity of the cost. Furthermore, beyond necessary conditions, it is crucial to find sufficient conditions that certify optimality. In our setting this extension is fairly intuitive, since it is needed to check that the second order derivative of the cost functional is positive around the optimal control in order to find local minimizers. This would introduce the second order adjoint equation where the coefficients would be the second order derivatives of the coefficients of the state equation and the functions f and h. This has already been proved in the classical framework of the PMP with deterministic final time.

1

Preliminaries on functional analysis and stochastic analysis

1.1 Functional Analysis

In this chapter, we recall some fundamental concepts from functional analysis that will be used throughout the thesis. The presentation does not aim to be exhaustive, but rather to collect the main concepts, spaces and inequalities that will be used later all of which have already been proven during classes. The concepts and results are, also, known and well presented in the literature, such as in Berger (1977), Rudin (2007) and Brezis (2011).

1.1.1 Normed, Banach and Hilbert Spaces

We begin by recalling the definition of a normed space. Let H be a vector space. A *norm* on H is a function $\|\cdot\| : H \to [0, \infty)$ such that:

- $\|x\| = 0$ if and only if $x = 0$;
- $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in H$ and all scalars α;
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in H$ (triangle inequality).

The tuple $(H, \|\cdot\|)$ is called a *normed space*. In a normed space, if we consider a sequence $\{x_n\}$ in H, we say that it is a *Cauchy sequence* if for every $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, we have $\|x_n - x_m\| < \epsilon$. A normed space is called a *Banach space* if every Cauchy sequence in the space converges to a limit that is also in the space.

We denote as *inner product space* when we consider H as a vector space with an inner product. When $(H, \langle\cdot,\, \cdot\rangle)$ is complete with respect to the norm induced by the inner product, i.e., $\|x\| = \sqrt{\langle x,\, x\rangle}$, then it is called a *Hilbert space*.

Let us fix some notation which is going to be useful later on. For $x \in \mathbb{R}^n$, we denote by $|x|$ its Euclidean norm, and by $\langle x, \bar{x}\rangle$ the standard inner product of $x, \bar{x} \in \mathbb{R}^n$. For matrices, we write $y \in \mathbb{R}^{n\times d}$ for a real-valued $n \times d$ matrix, and $y^T \in \mathbb{R}^{d\times n}$ for its transpose. The Euclidean (or Frobenius) norm of a matrix is defined by $|y| = \sqrt{\mathrm{Tr}(yy^T)}$ while the inner product of two matrices $y, \bar{y} \in \mathbb{R}^{n\times d}$ is given by $\langle y, \bar{y}\rangle = \sqrt{\mathrm{Tr}(y\bar{y}^T)}$

Now if we consider a measure space $(H, \mathcal{A}, \mu)$, we can define a special type of Banach spaces called *Lebesgue spaces* or L^p spaces. For $1 \leq p < \infty$, they are defined as

$$L^p(H, \mathcal{A}, \mu) = \left\{ f : H \to \overline{\mathbb{R}} :\ f \text{ is measurable and } \int_H |f|^p \,\mathrm{d}\mu < \infty \right\},$$

and define

$$L^\infty(H, \mathcal{A}, \mu) := \left\{ f : H \to \overline{\mathbb{R}} :\ f \text{ is measurable and ess sup } f < \infty \right\}.$$

A fundamental tool when working with product measure space is the possibility of interchanging the order of integration.

Theorem 1.1 (Fubini-Tonelli)**.** Consider the measure space $(X \times Y, \mathcal{A} \otimes \mathcal{K}, \mu \otimes \nu)$, where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{K}, \nu)$ are σ-finite measure spaces. Let f be a function $(\mathcal{A} \otimes \mathcal{B})$-measurable and defined everywhere on $X \times Y$.
If the function is non-negative, i.e. $0 \leq f \leq +\infty$, then $x \mapsto \int_Y f(x, y)\mathrm{d}\nu(y)$ is $\mathcal{B}(\bar{\mathbb{R}})$ measurable and $y \mapsto \int_X f(x, y)\mathrm{d}\mu(x)$ is $calB(\bar{\mathbb{R}})$ measurable.

Moreover, the following *iterated integral formula* holds:

$$\int_{X\times Y} f \mathrm{d}(\mu \otimes \nu) = \int_X \left(\int_Y f(x, y)\mathrm{d}\nu(y) \right) \mathrm{d}\mu(x) = \int_Y \left(\int_X f(x, y)\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y).$$

1.1.2 Operators

Let H_1 and H_2 be normed spaces. An *operator* $T : H_1 \to H_2$ is a mapping between them. Of particular interest are *linear and bounded* operators.

A linear operator is said to be *bounded* if there exists a constant $C > 0$, independent of x, such that for all $x \in H_1$, we have

$$\|T(x)\|_{H_2} \leq C\|x\|_{H_1}.$$

An operator is *continuous* in $x \in H_1$ if for every $\{x_n\} \subset H_1$, $x_n \to x$ we have that $T(x_n) \to T(x)$. It can be proved that a linear operator is bounded if and only if it is continuous. The set of all bounded linear operators from H_1 to H_2 is denoted by $\mathcal{L}(H_1, H_2)$ with respect to the operator norm:

$$\|T\|_{\mathcal{L}(H_1,H_2)} = \sup_{x \in H_1, x \neq 0} \frac{\|T(x)\|_{H_2}}{\|x\|_{H_1}}.$$

1.1.3 Functional Derivative

In order to study perturbations of operators and functionals, we require the notion of differentiability in Banach spaces. Two commonly used notions are the Fréchet derivative and the Gâteaux.

Definition. Let $x_0 \in H_1$, $x_0 \in U \subset H_1$ open and $T : U \to H_2$. T is *Fréchet differentiable* at x_0 if there is a linear operator $A \in \mathcal{L}(H_1, H_2)$ such that:

$$\|T(x) - T(x_0) - A[x - x_0]\|_{H_2} = o(\|x - x_0\|_{H_1})$$

In this case we write $DT(x_0) = A$ and $DT(x_0)$ is called the *Fréchet differential* of T at x_0.

If the mapping $x \mapsto DT(x)$ is continuous from H_1 to $\mathcal{L}(H_1, H_2)$, T is called C^1. The Fréchet differential is the natural extension of the classical derivative to Banach spaces. The Gâteaux derivative, instead, corresponds to directional differentiability along arbitrary directions.

Definition. Let $x_0 \in H_1$, $x_0 \in U \subset H_1$ open and $T : U \to H_2$. T is *Gâteaux* *differentiable* at x_0 if there is an operator $A \in \mathcal{L}(H_1, H_2)$, such that $\forall h \in H_1$

$$\lim_{\epsilon \to 0} \|T(x_0 + \epsilon h) - T(x_0) - \epsilon A[h]\|_{H_2} = 0$$

We write $A = DT(x_0)$ and we call it the *Gâteaux derivative* of T at x_0.

Although weaker, the Gâteaux differential often suffices in applications, particularly in variational analysis. Some obvious properties are:

1. The Fréchet differentials and Gâteaux differentials are unique.
2. If T is Gâteaux differentiable at $x_0 + \epsilon h, (0 \leq \epsilon \leq 1)$

$$T(x_0 + h) - T(x_0) = \int_0^1 DT(x_0 + \epsilon h)[h] d\epsilon.$$

Theorem 1.2. Let $x_0 \in H_1$, $x_0 \in U \subset H_1$ open and $T : U \to H_2$. If T is Fréchet differentiable at x_0, it is also Gâteaux differentiable at x_0. Conversely, if T is Gâteaux differentiable at x for every $x \in U$ and if $x \mapsto DT(x)$ is continuous as a map from H_1 to $\mathcal{L}(H_1, H_2)$, then T is Fréchet differentiable.

Rules for Fréchet differentiation are similar to the finite-dimensional case, in particular we have the *chain rule*. Suppose H_1, H_2, H_3 are Banach spaces, and $U \subset H_1$, $V \subset H_2$ are open sets. Then if $T : U \to V$ and $S : V \to H_3$,

$$D(S \circ T)(x) = DS(T(x)) \circ DT(x), \quad \forall x \in U.$$

We may find a proof of these results in Berger (1977) as well as a deeper analysis of the functional derivatives.

1.1.4 Useful Inequalities

We conclude this chapter by recalling several classical inequalities that will be useful in later proofs. These results can be found in Hardy et al. (1952) and Krylov (1980).

Theorem 1.3 (*Triangular inequality*)**.** Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space. For all $x, y \in H$, we have:

$$\|x + y\| \leq \|x\| + \|y\|.$$

Theorem 1.4 (*Young's inequality*)**.** Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. For all $x, y \in H$, we have:

$$|\langle x, y\rangle| \leq \frac{\lambda^2}{2}\|x\|^2 + \frac{1}{2\lambda^2}\|y\|^2.$$

Theorem 1.5 (*Cauchy-Schwarz inequality*)**.** Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space. For all $x, y \in H$, we have:

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

Theorem 1.6 (*Gronwall lemma*)**.** Let $f \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the integral inequality

$$f(t) \leq f_0 + \int_0^t g(s)f(s)ds + \int_0^t h(s)ds, \quad t \in [0,T),$$

for some $0 \leq g \in L^1(0,T)$ and $h \in L^1(0,T)$. Then, f satisfies the pointwise estimate

$$f(t) \leq e^{\Lambda(t)} + \int_0^t e^{\Lambda(t)-\Lambda(s)}h(s)ds, \quad t \in [0,T),$$

where $\Lambda(t) = \int_0^t g(s)ds$.

Some examples and important special cases of the Gronwall lemma are:

$$f(t) \leq f_0 + \int_0^t g(s)f(s)ds \Rightarrow f(t) \leq f_0 e^{\Lambda(t)},$$
$$f(t) + \int_0^t |h(s)|\, ds \leq f_0 + \int_0^t g(s)f(s)ds \Rightarrow f(t) + \int_0^t |h(s)|\, ds \leq f_0 e^{\Lambda(t)}.$$

1.2 Stochastic Analysis

In this chapter we recall some fundamental concepts of stochastic analysis that will be used throughout the thesis. Results in this sections are known in the literature, Baldi (2017), Øksendal (2003)

1.2.1 Elements of Probability

We must introduce some concepts from basic probability theory, which is extensively covered in the literature, such as in Baldi (2017), Jacod and Protter (2004).

A *measurable space* is a pair $(E, \mathcal{E})$ where

- E is a set;
- $\mathcal{E}$ is a σ-algebra of subsets of E.

A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is a set;
- $\mathcal{F}$ is a σ-algebra of subsets of Ω;
- $\mathbb{P}$ is a positive measure on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $(E, \mathcal{E})$ a measurable space, a *random variable* is a measurable function $X : \Omega \to E$, i.e. such that $X^{-1}(A) \in \mathcal{F}$ whenever $A \in \mathcal{E}$. It is a *real random variable* if $E = \mathbb{R}$ and $\mathcal{E} = \mathcal{B}(\mathbb{R})$ (Borel sets of $\mathbb{R}$).

If a real r.v. X is integrable, denote the *expectation* as

$$\mathbb{E}[X] := \int X \mathrm{d}\mathbb{P}.$$

Moreover, if X is an integrable real r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{D}$ a sub-σ-algebra of $\mathcal{F}$. The *conditional expectation* of X with respect to $\mathcal{D}$, written as $\mathbb{E}[X|\,\mathcal{D}]$ is the of r.v.'s Z such that:

1. Z is $\mathcal{D}$-measurable and integrable;
2. for all $D \in \mathcal{D}$
$$\int_D X \mathrm{d}\mathbb{P} = \int_D Z \mathrm{d}\mathbb{P}.$$

Note that a r.v. Z with these properties always exists and is unique up to $\mathbb{P}$-equivalence.

Proposition 1.1. Let X, X_1, X_2 be integrable r.v.'s and $\alpha, \beta \in \mathbb{R}$. Then

- $\mathbb{E}[\alpha X_1 + \beta X_2|\,\mathcal{D}] = \mathbb{E}[\alpha X_1|\,\mathcal{D}] + \mathbb{E}[\beta X_2|\,\mathcal{D}]$ $\mathbb{P}$-a.s.;
- *Tower property*: $\mathbb{E}[\mathbb{E}[X|\,\mathcal{D}]] = \mathbb{E}[X]$;
- If Z is bounded and $\mathcal{D}$-measurable then $\mathbb{E}[ZX|\,\mathcal{D}] = Z\mathbb{E}[X|\mathcal{D}]$ $\mathbb{P}$-a.s.;
- If X is independent of $\mathcal{D}$ then $\mathbb{E}[X|\,\mathcal{D}] = \mathbb{E}[X]$ $\mathbb{P}$-a.s.

1.2.2 Stochastic Processes

In the sequel, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space as defined above. For $p \in [1, \infty)$, we denote by $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$ the set of random variables ξ (valued in $\mathbb{R}^n$) such that $\mathbb{E}[|\xi|^p] < \infty$.

A stochastic process is a measurable map $X : \Omega \times [0, T] \to \mathbb{R}^n$, which may be seen equivalently as a family $X = (X_t)_{t\in[0,T]}$ of random variables valued in $\mathbb{R}^n$ equipped with its Borel σ-field and indexed by time t.

Definition. A *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{F}_t)_{t\in[0,T]}$ of σ-fields of $\mathcal{F}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t \leq T$.

We say that a filtration $(\mathcal{F}_t)$ satisfies the *usual conditions* if it is right-continuous, i.e.

$$\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t, \quad \forall t \in [0, T),$$

and if it is complete, i.e., $\mathcal{F}_0$ contains all the negligible sets of $\mathcal{F}$.

Definition. A process $(X_t)_{t\in[0,T]}$ is *adapted* (with respect to the filtration $(\mathcal{F}_t)$) if for all $t \in [0, T]$, X_t is $\mathcal{F}_t$-measurable.

Definition. A process $(X_t)_{t\in[0,T]}$ is said to be *progressively measurable* if for every $s \in [0, T]$ the map $(\omega, t) \mapsto X_t(\omega)$ is measurable $(\Omega \times [0, s], \mathcal{B}([0, s]) \otimes \mathcal{F}_s) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$.

A progressively measurable process is measurable. A subset $P \subseteq \Omega \times [0, \infty)$ is said to be *progressively measurable* if the process $X_s(\omega) := \mathbb{1}_P(\omega, s)$ is progressively measurable in the sense above, where $\mathbb{1}_P$ is the indicator function of P. The set of all such subsets P form a sigma algebra on $\Omega \times [0, \infty)$, denoted by $\mathcal{P}$.

Having in mind the interpretation of $\mathcal{F}_t$ as the available information up to time t, we want to know if an event characterized by its first arrival time $\tau(\omega)$, occurred or not before time t given the observation in $\mathcal{F}_t$. This leads to the notion of stopping time.

Definition. A random variable $\tau : \Omega \to [0, \infty]$ is a *stopping time* with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ if for all $t \in [0, T]$

$$\{\tau \leq t\} := \{\omega \in \Omega : \ \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

Associated to a stopping time τ let us define the *stopped σ-algebra*

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : \; A \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for every } t \in [0,T]\}.$$

It easily checked that any random time equal to a positive constant t is a stopping time. If the filtration satisfies the usual conditions, we also notice that if τ and σ are two stopping times, then $\tau \wedge \sigma$, $\tau \vee \sigma$ and $\tau + \sigma$ are stopping times.

Now we introduce the following notations, :

- $L^2_T(\mathbb{R}^l)$: the space of all $\mathcal{F}_T$-measurable $\mathbb{R}^l$-valued random variables such that

$$\|X\|^2_{L^2} := \mathbb{E}[|X|^2] < \infty.$$

- $\mathbb{S}^2_T(\mathbb{R}^l)$: the space of $\mathbb{R}^l$-valued continuous adapted processes Y such that

$$\|Y\|_{\mathbb{S}^2} := \mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right]^{\frac{1}{2}} < \infty.$$

- $\mathbb{H}^2_T(\mathbb{R}^l)$: the space of $\mathbb{R}^l$-valued progressively measurable processes Z such that

$$\|Z\|_{\mathbb{H}^2} := \left(\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} < \infty.$$

- For $\beta > 0$, we define a weighted norm on $\mathbb{H}^2_T(\mathbb{R}^l)$ by

$$\|X\|_{\mathbb{H}^2_\beta} := \left(\mathbb{E}\left[\int_0^T e^{\beta t}|X_t|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} < \infty,$$

- For a $\mathcal{F}$-stopping time τ, we define $L^2_\tau(\mathbb{R}^l)$ the space of all $\mathcal{F}_\tau$ measurable $\mathbb{R}^l$-valued random variables such that

$$\|X\|^2_{L^2_\tau} := \mathbb{E}[|X|^2] < \infty.$$

- For a $\mathcal{F}$-stopping time τ, $\mathbb{H}^2_\tau(\mathbb{R}^l)$ is the space of $\mathbb{R}^l$-valued progressively measurable processes Z such that

$$\|Z\|_{\mathbb{H}^2_\tau} := \left(\mathbb{E}\left[\int_0^\tau |Z_t|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} < \infty.$$

- For a $\mathcal{F}$-stopping time τ, and for $\beta > 0$, we define a weighted norm on $\mathbb{H}^2_\tau(\mathbb{R}^l)$ by

$$\|X\|_{\mathbb{H}^2_{\beta,\tau}} := \left(\mathbb{E}\left[\int_0^\tau e^{\beta t}|X_t|^2 \mathrm{d}t\right]\right)^{\frac{1}{2}} < \infty,$$

  and denote by $\mathbb{H}^2_{\beta,\tau}(\mathbb{R}^l)$ the space $\mathbb{H}^2_\tau(\mathbb{R}^l)$ endowed with the norm $\|\cdot\|_{\mathbb{H}^2_\beta}$.

1.2.3 Stochastic Calculus

Let I_d be the d-dimensional identity matrix. The main example of a process is Brownian motion.

Definition. A *standard d-dimensional Brownian motion* on $[0, T]$ is a continuous process valued in $\mathbb{R}^d$, $(W_t)_{t\in[0,T]} = (W_t^1, \dots, W_t^d)_{t\in[0,T]}$ such that:

1. $W_0 = 0$.
2. For all $0 \le s < t \le T$, the increment $W_t - W_s$ is independent of $\sigma(W_u,\, u \le s)$ and follows a centered Gaussian distribution with variance-covariance matrix $(t - s)I_d$.

A major problem concerns the existence and construction of a Brownian motion. We do not discuss this problem here, and refer to Karatzas and Shreve (1998). We only state some classical properties of a Brownian motion.

Theorem 1.7. Let $(W_t)_{t\in[0,T]}$ be a Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$:

1. *Symmetry*: $(-W_t)_{t\in[0,T]}$ is also a Brownian motion.
2. *Scaling*: for all $\lambda > 0$, the process $((1/\lambda)W_{\lambda^2 t})_{t\in[0,T]}$ is also a Brownian motion.
3. *Invariance by translation*: for all $s > 0$, the process $(W_{t+s} - W_s)_{t\in[0,T]}$ is a standard Brownian motion independent of $\mathcal{F}_s$.

Martingales are central in stochastic analysis, as they capture the idea of processes with no drift when conditioned on past information.

Definition. An adapted process $(X_t)_{t\in[0,T]}$ is a *martingale* if, for all $0 \le s < t \le T$, $X_t \in L^1(\Omega, \mathcal{F}_t)$ and we also have:

$$\mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s.$$

Definition. A process $(X_t)_{t\in[0,T]}$ is a *local martingale* if there exists an increasing sequence $(\tau_n)_n$ of stopping times such that:

1. $\tau_n \nearrow +\infty$ $\mathbb{P}$-a.s. for $n \to \infty$;
2. $(X_{t\wedge\tau})_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale for every n.

We state a first fundamental inequality for martingales which proof can be found in Baldi (2017).

Theorem 1.8 (Doob's inequality)**.** Let $X = (X_t)_{t\in[0,T]}$ be a non-negative martingale. Then for any stopping time τ valued in $[0, T]$, we have:

$$\mathbb{E}\left[\sup_{0\le t\le\tau} |X_t|\right]^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_\tau|^p], \quad p > 1.$$

Theorem 1.9. Let $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ be two square integrable and continuous local martingale with respect to a filtration $(\mathcal{F}_t)_t$, augmented with the negligible events of $\mathcal{F}$. Then there exists a unique adapted continuous process of finite variation, denoted by $[X, Y]$ called *quadratic covariation*, such that $XY - [X, Y]$ is a local martingale. Moreover, for all $t \in [0, T]$, if $0 < t_0^n < t_1^n < ... < t_{k_n}^n = t$ is a subdivision of $[0, t]$ with mesh size going to 0, then we have:

$$[X, Y]_t = \lim_{n\to+\infty} \sum_{i=1}^{k_n} (X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_i^n} - Y_{t_{i-1}^n}),$$

for the convergence in probability.

When $X = Y$, the process $[X, X]$, also denoted by $[X]$ or $\langle X\rangle$ and called the *quadratic variation* of X.

The following is a very important inequality which is proved in Dellacherie (1975)

Theorem 1.10 (Burkholder-Davis-Gundy inequality)**.** For all $p > 0$, there exist positive constants c_p and C_p such that for all continuous local martingales $M = (M_t)_{t\in[0,T]}$ and all stopping times τ valued in $[0, T]$, we have:

$$c_p\mathbb{E}\left[\langle M\rangle_\tau^{\frac{p}{2}}\right] \le \mathbb{E}\left[\sup_{0\le t\le\tau} |M_t|\right]^p \le C_p\mathbb{E}\left[\langle M\rangle_\tau^{\frac{p}{2}}\right].$$

We are now ready to introduce the Itô calculus which will be fundamental later on, the proofs of the following theorems can be found, for example, in Baldi (2017) or in Karatzas and Shreve (1998)

Definition. Let $W = (W^1, ..., W^d)$ be a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We define an *Itô process* as a process $X = (X^1, ..., X^n)$ valued in $\mathbb{R}^n$ such that a.s.

$$X_t = X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s, \quad t \in [0, T],$$

where X_0 is $\mathcal{F}_0$-measurable and $b \in L^1(0, T; \mathbb{R}^n)$ and $\sigma \in L^2(0, T; \mathbb{R}^{n\times d})$ $\mathbb{P}$-a.s.

Theorem 1.11 (Itô's formula)**.** Let $X = (X_t)_{t\in[0,T]}$ be an Itô process valued in $\mathbb{R}^n$ and f a function of class $C^{1,2}([0, T] \times \mathbb{R}^n)$. Then the process $Y = (Y_t)_{t\in[0,T]}$ defined by $Y_t = f(t, X_t)$ is also an Itô process and $\mathbb{P}$-a.s.

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f}{\partial s}(s, X_s)\mathrm{d}s + \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)\mathrm{d}X_s^i + \\ + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)\mathrm{d}\langle X^i, X^j\rangle_s.$$

Theorem 1.12 (Martingale Representation theorem)**.** Assume that $\mathcal{F}$ is the natural filtration (augmented) of a standard d-dimensional Brownian motion $W = (W^1, ..., W^d)$. Let $X = (X_t)_{t\in[0,T]}$ be a square integrable martingale. Then there exists $\alpha = (\alpha^1, ..., \alpha^d) \in L^2(\Omega; L^2(0, T; \mathbb{R}^d))$ progressively measurable such that $\mathbb{P}$-a.s.

$$X_t = X_0 + \sum_{i=1}^{d} \int_0^t \alpha_s^i \mathrm{d}W_s^i, \quad t \in [0, T].$$

Note that we will always identify the martingale X by its continuous version.

1.2.4 Stochastic Differential Equations

We recall in this section some results about stochastic differential equations (SDE) with respect to a Brownian motion.

We fix a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions and a d-dimensional Brownian motion $W = (W^1, ..., W^d)$ with respect to $(\mathcal{F}_t)$. We are given functions

$$b : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$\sigma : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$$

with $b(\omega,t,x) = (b_i(\omega,t,x))_{1\le i\le d}$, $\sigma(\omega,t,x) = (\sigma_{ij}(\omega,t,x))_{1\le i\le n, 1\le j\le d}$. We assume that for all ω, the functions $b(\omega,\cdot,\cdot)$ and $\sigma(\omega,\cdot,\cdot)$ are Borelian on $[0,T]\times\mathbb{R}^n$ and that for all $x\in\mathbb{R}^n$, the processes $b(\cdot,x,\cdot)$ and $\sigma(\cdot,x,\cdot)$ are progressively measurable with respect to $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^{n\times d})$. We then consider the *stochastic differential equation* (SDE):

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t, \quad t\in[0,T], \quad X_0 = x_0. \tag{SDE}$$

Definition. A *strong solution* of the (SDE) starting at time $t = 0$ is a vectorial progressively measurable process $X = (X^1, ...X^n)$ such that

$$\int_0^T |b(u,X_u)|\mathrm{d}u + \int_0^T |\sigma(u,X_u)|^2\mathrm{d}u < \infty, \quad \mathbb{P}\text{-a.s.},$$

and

$$X_s = x_0 + \int_0^s b(u,X_u)\mathrm{d}u + \int_0^s \sigma(u,X_u)\mathrm{d}W_u, \quad \forall s\in[0,T] \quad \mathbb{P}\text{-a.s.}$$

Note that if we include the probability space into the solution of the SDE, we call it a *weak solution*.

Definition. We say that for the (SDE) there is *pathwise uniqueness* if, given two solutions X^1 and X^2 defined on the same probability space and with respect to the same Brownian motion, X^1 and X^2 are indistinguishable, i.e.

$$\mathbb{P}(X_t^1 = X_t^2 \text{ for every } t\in[0,T]) = 1.$$

Existence and uniqueness of a strong solution to the SDE is ensured by the following Lipschitz and linear growth conditions.

Assumptions ($\mathcal{H}$-1). *For every $\omega \in \Omega$ the map $(t,x) \mapsto b(\omega,t,x)$ is measurable with respect to $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^n)$ and similarly for σ. Moreover, there exist constants $M > 0$, $L > 0$ such that for all $(t,x),(t,y) \in [0,T] \times \mathbb{R}^n$ we have:*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|, \quad \mathbb{P}\text{-}a.s.$$
$$|b(t,x)| + |\sigma(t,x)| \leq M(1+|x|), \quad \mathbb{P}\text{-}a.s.$$

Theorem 1.13 (Existence and Uniqueness)**.** Under Assumptions ($\mathcal{H}$-1), there exists for all $t \in [0,T]$, a strong solution to the (SDE). Moreover, for any ξ $\mathcal{F}_0$-measurable random variable valued in $\mathbb{R}^n$, such that $\mathbb{E}\left[|\xi|^p\right] < \infty$, for some $p > 1$, there is pathwise uniqueness of a strong solution X starting from ξ at time 0. This solution is p-integrable: for all $T > 0$, there exists a constant C_T such that

$$\mathbb{E}\left[\sup_{s \in [t,T]} |X_s|^p\right] \leq C_T \left(1 + \mathbb{E}\left[|\xi|^p\right]\right).$$

This result is standard and can be found in Baldi (2017).

Theorem 1.14. There exists a constant C such that if ξ and ξ' are two r.v. $\mathcal{F}_0$-measurable and both in $L^p(\mathbb{R}^n)$, X and X' be the corresponding unique solution with initial data ξ and ξ' respectively. Then

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |X_s - X'_s|^p\right] \leq C|\xi - \xi'|^p.$$

2

Backward Stochastic Differential Equations

The theory of backward stochastic differential equations (BSDEs) was first introduced by Pardoux (1995). Since then, it has become an important field of research due to its connections with stochastic control, mathematical finance, and partial differential equations.

In the case of an ordinary differential equation (ODE), under the usual Lipschitz conditions, both the initial value and terminal value problems are well-posed. In fact, on the interval $[0,T]$, a terminal value problem for an ODE is equivalent to an initial value problem on $[0,T]$ under the time reversal $t \mapsto T-t$. The situation is, however, fundamentally different in the stochastic framework when we are looking for a solution that is adapted to the given filtration. A naïve time reversal would, therefore, destroy the adaptiveness of the solution.

This difficulty can be seen in the simplest possible BSDE on $[0,T]$, with terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^n)$ as explained in Yong and Zhou (1999):

$$\begin{cases} \mathrm{d}Y_t = 0, & t \in [0,T] \\ Y_T = \xi. \end{cases} \tag{2.1}$$

The obvious unique solution is given by

$$Y_t = \xi, \quad t \in [0,T], \tag{2.2}$$

which is not necessarily $\mathcal{F}_t$-adapted (unless ξ is $\mathcal{F}_0$-measurable, i.e. deterministic). Hence, in its raw form the BSDE is ill-posed if one insists on the existence of an adapted solution.

A natural approach is to reformulate for the (2.1) while keeping most features of it. In particular, it should coincide with it when ξ is a nonrandom constant since in that case, it does have an adapted solution given by (2.2). However, this definition alone does not satisfy the original BSDE (2.1) dynamics. A first attempt is to set

$$Y_t = \mathbb{E}[\xi|\mathcal{F}_t], \quad t \in [0,T].$$

Then, Y is clearly $\mathcal{F}_t$-adapted and satisfies the terminal condition $Y_T = \xi$, and reduces to the constant solution $Y_t = \mathbb{E}[\xi|\mathcal{F}_t] = \xi, t \in [0,T]$ when ξ is a nonrandom constant. However, Y no longer satisfies (2.1).

The missing ingredient comes from the martingale representation Theorem 1.12, which guarantees the existence of a process $Z \in L^2(\Omega, \mathcal{F}, \mathbb{R}^n)$ such that

$$\begin{cases} \mathrm{d}Y_t = Z_t \mathrm{d}W_t, & \forall t \in [0,T] \\ Y_T = \xi \end{cases} \tag{2.3}$$

Therefore the new part "corrects" the nonadaptiveness of (2.1): it compensates for the backward nature of the problem and restore adaptiveness.

Thus, the appropriate notion of $\mathcal{F}_t$-adapted solution is not a single adapted process, but rather a pair of adapted processes (X, Z) satisfying (2.3).

2.1 Existence and Uniqueness with deterministic terminal time

We follow the notation of El Karoui et al. (1997). Let $W = (W_t)_{t\in[0,T]}$ be a standard d-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the natural filtration of W augmented with the $\mathbb{P}$-negligible events, and $T > 0$ is a fixed finite horizon.

Let $\xi : \Omega \to \mathbb{R}^n$ be an $\mathcal{F}_T$-measurable random variable, and let

$$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$$

be a measurable function with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\mathbb{R}^{n\times d})$. A *backward stochastic differential equation* (BSDE) with *terminal value* ξ and *generator* f is given by

$$\mathrm{d}Y_t = -f(t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \quad Y_T = \xi$$

or equivalently, in integral form,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s, \quad t \in [0,T]. \tag{BSDE}$$

Definition. A *square-integrable solution* to the (BSDE) is a pair of processes $(Y, Z) \in \mathbb{S}^2_T(\mathbb{R}^n) \times \mathbb{H}^2_T(\mathbb{R}^{n\times d})$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s, \quad t \in [0,T], \quad \mathbb{P}\text{-a.s.}.$$

The theory on the solutions of BSDEs is based on the following conditions.

Assumptions ($\mathcal{H}$-2). *The pair $\xi : \Omega \to \mathbb{R}^n$, $\mathcal{F}_T$-measurable random variable and*

$$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$$

satisfy:

1. $\xi \in L^2_T(\mathbb{R}^n)$.
2. $\{f(\cdot,0,0),\ t \in [0,T]\} \in \mathbb{H}^2_T(\mathbb{R}^n)$.
3. *There exists a constant* $C > 0$ *such that*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|)$$

holds true for all $t \in [0,T]$, $(y_1,z_1),(y_2,z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, $\mathbb{P}$*-a.s.*

In order to prove the existence and uniqueness of solutions to (BSDE), we need the following fundamental result from Yong and Zhou (1999).

Proposition 2.1 (A priori estimate)**.** Let (f^1,ξ^1) and (f^2,ξ^2) satisfy Assumptions ($\mathcal{H}$-2), and let (Y^1,Z^1) and (Y^2,Z^2) be any corresponding square-integrable solutions, respectively, define,

$$\begin{aligned}\delta Y &:= Y^1 - Y^2,\\ \delta Z &:= Z^1 - Z^2,\\ \delta_2 f &:= f^1(\cdot,Y^2,Z^2) - f^2(\cdot,Y^2,Z^2).\end{aligned}$$

Let $C_1 > 0$ be the Lipschitz constant of f^1. Then, for any $(\lambda,\mu,\beta) \in \mathbb{R}^3$ such that $\mu > 0$, $\lambda^2 > C_1$ and $\beta \ge C_1(2+\lambda^2) + \mu^2$ it holds

$$\begin{aligned}\|\delta Y\|^2_{\mathbb{H}^2_\beta} &\le T\left(e^{\beta T}\mathbb{E}\left[|\delta Y_T|^2\right] + \frac{1}{\mu^2}\|\delta_2 f\|^2_{\mathbb{H}^2_\beta}\right)\\ \|\delta Z\|^2_{\mathbb{H}^2_\beta} &\le \frac{\lambda^2}{\lambda^2 - C_1}\left(e^{\beta T}\mathbb{E}\left[|\delta Y_T|^2\right] + \frac{1}{\mu^2}\|\delta_2 f\|^2_{\mathbb{H}^2_\beta}\right).\end{aligned}$$

Proof. The proof is divided in 5 steps:

1. Note that since Y is a solution to (BSDE) in the sense of Definition 2.1 then

$$\sup_{0\le t\le T} |Y_t| \in L^2_T(\mathbb{R}^n).$$

2. Note that $\mathbb{P}$-a.s. for $t \in [0,T]$

$$\delta Y_t = \xi^1 - \xi^2 + \int_t^T \left(f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)\right) \mathrm{d}s - \int_t^T \left(Z^1_s - Z^2_s\right) \mathrm{d}W_s.$$

Applying Itô's formula 1.11 to $e^{\beta t} |\delta Y_t|^2$ from t to T yields:

$$\begin{aligned} e^{\beta t} |\delta Y_t|^2 = e^{\beta T} |\delta Y_T|^2 + 2\int_t^T e^{\beta s} \langle \delta Y_s, f^1(s, Y^1_s, Z^1_s) - f^2(s, Y^2_s, Z^2_s)\rangle \mathrm{d}s \\ - 2\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s \mathrm{d}W_s\rangle - \int_t^T e^{\beta s} |\delta Z_s|^2 \mathrm{d}s - \beta \int_t^T e^{\beta s} |\delta Y_s|^2 \mathrm{d}s. \end{aligned}$$

3. Our next goal is to show that

$$\mathbb{E}\left[e^{\beta t} |\delta Y_t|^2\right] \le e^{\beta T}\mathbb{E}\left[|\delta Y_T|^2\right] + \frac{1}{\mu^2}\mathbb{E}\left[\int_t^T e^{\beta s} |\delta_2 f_s|^2 \mathrm{d}s\right]$$

holds true for $t \in [0,T]$.
For $\sup_{0\le t\le T} |\delta Y_t| \in L^2_T(\mathbb{R})$, we conclude that

$$\begin{aligned} \mathbb{E}\left[\sqrt{\int_0^T |e^{\beta s}\delta Z_s \delta Y_s|^2 \mathrm{d}s}\right] &\le \mathbb{E}\left[\sqrt{\int_0^T e^{2\beta s} |\delta Z_s|^2 |\delta Y_s|^2 \mathrm{d}s}\right] \\ &\le \mathbb{E}\left[\sup_{0\le t\le T} |\delta Y_s| \sqrt{\int_0^T e^{2\beta s} |\delta Z_s|^2 \mathrm{d}s}\right] \\ &\le \sqrt{\mathbb{E}\left[\sup_{0\le t\le T} |\delta Y_s|^2\right]} \underbrace{\sqrt{\mathbb{E}\left[\int_0^T e^{2\beta s} |\delta Z_s|^2 \mathrm{d}s\right]}}_{Z\in\mathbb{H}^2_{T,\beta}} \\ &< \infty. \end{aligned}$$

Thus $e^{\beta s}\delta Z_s \delta Y_s \in \mathbb{H}^1_T$ which implies that the expectation of the stochastic integral $\int_t^T e^{\beta s}\langle \delta Y_s, \delta Z_s \mathrm{d}W_s\rangle$ is zero. Taking expectations, we obtain

$$\mathbb{E}\left[e^{\beta t}\left|\delta Y_t\right|^2\right] = e^{\beta T}\mathbb{E}\left[\left|\delta Y_T\right|^2\right] + \mathbb{E}\left[\int_t^T 2e^{\beta s}\langle \delta Y_s, f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\rangle \mathrm{d}s\right]$$
$$- \mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta Z_s\right|^2 \mathrm{d}s\right] - \beta\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta Y_s\right|^2 \mathrm{d}s\right].$$

By Lipschitz continuity of f^1,

$$\left|f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2)\right| \leq C_1(|\delta Y_t| + |\delta Z_t|) + |\delta_2 f_t|.$$

We notice that, for positive, real numbers $\lambda, \mu > 0$ and $C, y, z, , t \geq 0$, the following inequality holds:

$$2y(Cz + t) \leq C\frac{z^2}{\lambda^2} + \frac{t^2}{\mu^2} + y^2(\mu^2 + C\lambda^2).$$

Indeed:

$$\begin{aligned} 0 &\leq \left(\sqrt{C}\frac{z}{\lambda} - \lambda y\sqrt{C}\right)^2 + \left(\frac{t}{\mu} - y\mu\right)^2 \\ &= C\frac{z^2}{\lambda^2} - 2yCz + \lambda^2 y^2 C + \frac{t^2}{\mu^2} - 2yt + y^2\mu^2 \\ &= C\frac{z^2}{\lambda^2} + \frac{t^2}{\mu^2} + y^2(\mu^2 + C\lambda^2) - 2y(Cz + t). \end{aligned}$$

By taking this into account, We infer:

$$\begin{aligned} &\mathbb{E}\left[\int_t^T 2e^{\beta s}\langle \delta Y_s, f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\rangle \mathrm{d}s\right] \\ \leq &\mathbb{E}\left[\int_t^T 2e^{\beta s}\left|\delta Y_s\right| \left|f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)\right| \mathrm{d}s\right] \\ \leq &\mathbb{E}\left[\int_t^T 2Ce^{\beta s}\left|\delta Y_s\right|^2 + e^{\beta s}2\left|\delta Y_s\right|(C_1\left|\delta Z_s\right| + |\delta_2 f_s|)\mathrm{d}s\right] \\ \leq &\left(C_1(2+\lambda^2) + \mu^2\right)\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta Y_s\right|^2 \mathrm{d}s\right] + \frac{C_1}{\lambda^2}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta Z_s\right|^2 \mathrm{d}s\right] + \\ &+ \frac{1}{\mu^2}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta_2 f_s\right|^2 \mathrm{d}s\right]. \end{aligned}$$

Choosing β, λ sufficiently large allows the negative terms to absorb the positive ones, so that

$$\mathbb{E}\left[e^{\beta t}\left|\delta Y_t\right|^2\right] \leq \mathbb{E}\left[e^{\beta T}\left|\delta Y_T\right|^2\right] + \frac{1}{\mu^2}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta_2 f_s\right|^2 \mathrm{d}s\right].$$

4. Integrating last inequality in t from 0 to T and using Fubini's Theorem 1.1 we obtain

$$\|\delta Y\|_{\mathbb{H}^2_\beta}^2 \leq T\left(e^{\beta T}\mathbb{E}\left[\left|\delta Y_T\right|^2\right] + \frac{1}{\mu^2}\|\delta_2 f\|_{\mathbb{H}^2_\beta}^2\right).$$

5. Finally, the estimate in point 3 also gives us

$$\left(1 - \frac{C_1}{\lambda^2}\right)\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta Z_s\right|^2 \mathrm{d}s\right] \leq e^{\beta T}\mathbb{E}\left[\left|\delta Y_T\right|^2\right] + \frac{1}{\mu^2}\mathbb{E}\left[\int_t^T e^{\beta s}\left|\delta_2 f_s\right|^2 \mathrm{d}s\right].$$

Setting $t = 0$ yields the desired bound on the spread of the processes Z^1 and Z^2 and concludes the proof.

□

The previous proposition paves allows us to state the main result of this section originally proved by Pardoux (1995).

Theorem 2.1 (Existence and Uniqueness)**.** Let (f, ξ) satisfy Assumptions ($\mathcal{H}$-**2**). Then, the BSDE (BSDE) admits a unique square-integrable solution $(Y, Z) \in \mathbb{S}^2_T(\mathbb{R}^n) \times \mathbb{H}^2_T(\mathbb{R}^{n\times d})$ in the sense of Definition 2.1.

Proof. Since $\mathbb{H}^2_{T,\beta}(\mathbb{R}^n)$ is a Banach space, we apply the Banach fixed point theorem to a suitable mapping. Define

$$\begin{aligned} \varphi : \mathbb{H}^2_{T,\beta}(\mathbb{R}^n) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d}) &\to \mathbb{H}^2_{T,\beta}(\mathbb{R}^n) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d}) \\ (y, z) &\mapsto (Y, Z) \end{aligned}$$

where (Y, Z) solves the BSDE with generator $f(t, y_t, z_t)$ and terminal value ξ, i.e.

$$Y_t = \xi + \int_t^T f(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \quad t \in [0, T].$$

Again, we divide the proof into two steps.

1. In this step we will verify that the mapping φ is well-defined. To this end, note that $f(\cdot, y_t, z_t)$ is in $\mathbb{H}^2_T(\mathbb{R}^n)$ as shown before in the a-priori estimate. Moreover, we have for $t \in [0,T]$ thanks to Cauchy-Schwarz 1.5

$$\mathbb{E}\left[\left|\int_t^T f(s, y_s, z_s)\mathrm{d}s\right|^2\right] \leq (T-t)\mathbb{E}\left[\int_t^T |f(s, y_s, z_s)|^2 \mathrm{d}s\right]$$
$$\leq T\mathbb{E}\left[\int_0^T |f(s, y_s, z_s)|^2 \mathrm{d}s\right] < \infty.$$

Then $\int_t^T f(s, y_s, z_s)\mathrm{d}s \in L^2_T(\mathbb{R}^n)$ for every $t \in [0,T]$. Thanks to this we can say that

$$M_t := \mathbb{E}\left[\int_0^T f(s, y_s, z_s)\mathrm{d}s + \xi \middle| \mathcal{F}_t\right],$$

is a square-integrable martingale. By the martingale representation Theorem 1.12, there exists a unique process $Z \in \mathbb{H}^2_T(\mathbb{R}^{n\times d})$ such that

$$M_t = M_0 + \int_0^t Z_s \mathrm{d}W_s, \quad t \in [0,T].$$

Now define

$$Y_t := M_t - \int_0^t f(s, y_s, z_s)\mathrm{d}s, \quad t \in [0,T].$$

Then Y is adapted, continuous, and satisfies

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s, y_s, z_s)\mathrm{d}s \middle| \mathcal{F}_t\right].$$

By Fubini's Theorem 1.1, the tower property and by hypothesis we get

$$\mathbb{E}\left[\int_0^T |Y_t|^2 \mathrm{d}t\right] \leq \int_0^T \mathbb{E}\left[\left|\int_t^T f(s, y_s, z_s)\mathrm{d}s + \xi\right|^2\right]\mathrm{d}t$$
$$\leq 2\int_0^T \mathbb{E}\left[\left|\int_t^T f(s, y_s, z_s)\mathrm{d}s + \xi\right|^2\right]\mathrm{d}t$$
$$< \infty.$$

In order to conclude the step, we need to clarify that the uniquely determined pair (Y, Z) solves the equation. But

$$\xi = Y_T = M_T - \int_0^T f(s, y_s, z_s)\mathrm{d}s$$
$$= M_0 + \int_0^T Z_s \mathrm{d}W_s - \int_0^T f(s, y_s, z_s)\mathrm{d}s.$$

Implies that Y_t satisfies the BSDE.

2. We now want to show that the map φ is a contraction, i.e.

$$\|\delta Y\|^2_{\mathbb{H}^2_\beta} + \|\delta Z\|^2_{\mathbb{H}^2_\beta} \leq K \left(\|\delta y\|^2_{\mathbb{H}^2_\beta} + \|\delta z\|^2_{\mathbb{H}^2_\beta} \right),$$

with $K < 1$. Indeed, if this is the case then there exists a unique pair $(Y, Z) \in \mathbb{H}^2_{T,\beta}(\mathbb{R}^n) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d})$ such that

$$(Y, Z) = \varphi(Y, Z)$$

and so (Y, Z) would be a solution to the BSDE (BSDE).
Take two inputs $(y^1, z^1), (y^2, z^2) \in \mathbb{H}^2_{T,\beta}(\mathbb{R}^n) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d})$ with respective images (Y^1, Z^1) and (Y^2, Z^2). Using the terminology of Proposition 2.1, the term $\delta_2 f_t$ now reads as

$$\delta_2 f_t = f(t, y^1_t, z^1_t) - f(t, y^2_t, z^2_t),$$

hence it is independent of Y and Z. The Lipschitz constant of the Proposition 2.1 is 0.
By Lipschitz continuity of f,

$$|\delta_2 f_t| \leq C_1(|\delta y_t| + |\delta z_t|).$$

Applying the Proposition 2.1 with $\mu = \sqrt{\beta}$ and for every $\lambda > 0$ gives

$$\|\delta Y\|^2_{\mathbb{H}^2_\beta} \leq 2\frac{T}{\beta} C_1^2 \left(\|\delta y\|^2_{\mathbb{H}^2_\beta} + \|\delta z\|^2_{\mathbb{H}^2_\beta} \right)$$
$$\|\delta Z\|^2_{\mathbb{H}^2_\beta} \leq 2\frac{1}{\mathbb{H}^2_\beta} C_1^2 \left(\|\delta y\|^2_{\mathbb{H}^2_\beta} + \|\delta z\|^2_{\mathbb{H}^2_\beta} \right).$$

Hence,

$$\|\delta Y\|^2_{\mathbb{H}^2_\beta} + \|\delta Z\|^2_{\mathbb{H}^2_\beta} \leq \frac{2(T+1)}{\beta} C_1^2 \left(\|\delta y\|^2_{\mathbb{H}^2_\beta} + \|\delta z\|^2_{\mathbb{H}^2_\beta} \right).$$

Choosing $\beta > 2(T+1)C^2$, the mapping φ becomes a contraction. Thus, there exists a unique fixed point $(\overline{Y}, \overline{Z}) \in \mathbb{H}^2_{\beta,T}(\mathbb{R}^n) \times \mathbb{H}^2_{\beta,T}(\mathbb{R}^{n\times d})$ satisfying

$$\overline{Y}_t = \xi + \int_t^T f(s, \overline{Y}_s, \overline{Z}_s)\mathrm{d}s - \int_t^T \overline{Z}_s \mathrm{d}W_s, \quad t \in [0,T].$$

□

Note that we can choose the continuous version Y defined by

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s, \overline{Y}_s, \overline{Z}_s)\mathrm{d}s \middle| \mathcal{F}_t \right],$$

i.e. $(Y, \overline{Z})$ is the unique solution of the BSDE.

2.2 Existence and Uniqueness with random terminal time

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space where $(W_t)_{t\geq 0}$ is a d-dimensional standard Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ is the augmented natural filtration of W.

Fix a time horizon $T > 0$ and let τ be a stopping time with values in $[0,T]$. We consider the following BSDE on the random time interval $[0,\tau]$:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s, \quad t \geq 0 \qquad \text{(rBSDE)}$$

where ξ is an $\mathcal{F}_\tau$-measurable random variable and $f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^{n\times d})$-measurable.

Definition. A pair of processes $(Y, Z) \in \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ is called a *square-integrable solution* of (rBSDE) if the following hold:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s, \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

Assumptions ($\mathcal{H}$-3). *The triplet* (f, ξ, τ) *satisfies:*

1. $\xi \in L^2_\tau(\mathbb{R}^n)$.
2. *For any* $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$ *the process* $(f(t, y, z))_{t\geq 0}$ *is* $(\mathcal{F}_t)_{t\geq 0}$*-progressively measurable and s.t.*

$$\mathbb{E}\left[\int_0^\tau |f(t, y, z)|^2 \, \mathrm{d}t\right] < +\infty.$$

3. f *is Lipschitz in* (y, z)*, i.e. for all* $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$ *there exists a constant* $C > 0$ *s.t.*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|) \quad \forall t \in [0, T], \quad \mathbb{P}\text{-a.s.}$$

The following theorem is the extension of the existence and uniqueness result to the random horizon case.

Theorem 2.2 (Existence and Uniqueness with random horizon)**.** Let (f, ξ, τ) satisfy Assumption ($\mathcal{H}$-**3**). Then, there exists a unique pair of processes $(Y, Z) \in \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ which solves the BSDE with random terminal time (rBSDE) in the sense of Definition 2.2.

Proof. The proof follows the same scheme as Theorem 2.1 with adaptations due to the random horizon.

We again want to apply the Banach fixed point theorem to the function:

$$\begin{aligned} \varphi : \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d}) &\to \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d}) \\ (y, z) &\mapsto (Y, Z) \end{aligned}$$

where (Y, Z) solves the (rBSDE) with generator $f(t, y_t, z_t)$ and terminal value ξ, i.e.

$$Y_t = \xi + \int_{t\wedge\tau}^\tau f(s, y_s, z_s)\mathrm{d}s - \int_{t\wedge\tau}^\tau z_s \mathrm{d}W_s, \quad \forall t \geq 0,$$

1. Let $(y, z) \in \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$. By assumptions ($\mathcal{H}$-3) on f,

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^\tau f(s, y_s, z_s)\mathrm{d}s\right|^2\right] &\leq \mathbb{E}\left[\left(\int_0^\tau |f(s, y_s, z_s)|\,\mathrm{d}s\right)^2\right] \\
&\leq \mathbb{E}\left[\left(\int_0^\tau (|f(s,0,0)| + C(|y_s| + |z_s|))\,\mathrm{d}s\right)^2\right] \\
&\leq 4\mathbb{E}\left[\left(\int_0^\tau f(s,0,0)\mathrm{d}s\right)^2\right] + \underbrace{4\mathbb{E}\left[L^2\left(\int_0^\tau |y_s|\,\mathrm{d}s\right)^2\right]}_{\leq C^1\|y\|^2_{\mathbb{H}^2_\tau}<\infty} \\
&\qquad + \underbrace{4\mathbb{E}\left[C^2\left(\int_0^\tau |z_s|\,ds\right)^2\right]}_{\leq C^2\|z\|^2_{\mathbb{H}^2_\tau}<\infty} \\
&< \infty
\end{aligned}$$

So, define the square-integrable martingale:

$$M_t := \mathbb{E}\left[\xi + \int_0^\tau f(s, y_s, z_s)\mathrm{d}s \,\middle|\, \mathcal{F}_t\right].$$

By the martingale representation Theorem 1.12, there exists a unique process $Z \in \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ such that

$$M_t = M_0 + \int_0^t Z_s \mathrm{d}W_s, \quad t \in [0, T].$$

Now define the adapted and continuous process Y via

$$\begin{aligned}
Y_t &:= \mathbb{E}\left[M_t - \int_0^t f(s, y_s, z_s)\mathrm{d}s\right] \mathbb{1}_{\{t\leq\tau\}} + \xi\mathbb{1}_{\{t>\tau\}} \\
&= \mathbb{E}\left[\xi - \int_t^\tau f(s, y_s, z_s)\mathrm{d}s\right] \mathbb{1}_{\{t\leq\tau\}} + \xi\mathbb{1}_{\{t>\tau\}}
\end{aligned}$$

By substituting the equation for M_t into this definition we get exactly the (rBSDE). Then $(Y, Z) \in \mathbb{H}^2_\tau(\mathbb{R}^n)\times\mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ solve the (rBSDE) with generator $f(\cdot, y, z)$ and terminal condition ξ. Thus the φ is well-defined.

2. Let us prove that φ is a contraction. Let $(y^1, z^1), (y^2, z^2) \in \mathbb{S}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$

and $(Y^1, Z^1) := \varphi(y^1, z^1)$ and $(Y^2, Z^2) =: \varphi(y^2, z^2)$.
Set

$$\begin{aligned}\delta Y &:= Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2,\\ \delta y &:= y^1 - y^2, \quad \delta z := z^1 - z^2,\\ \delta f &:= f(\cdot, y^1, z^1) - f(\cdot, y^2, z^2).\end{aligned}$$

Applying Itô's formula 1.11 to $e^{\beta t}|\delta Y_t|^2$ yields to:

$$\begin{aligned}&e^{\beta t}|\delta Y_t|^2 + \int_t^\tau e^{\beta s}|\delta Z_s|^2 \mathrm{d}s + \beta \int_t^\tau e^{\beta s}|\delta Y_s|^2 \mathrm{d}s\\ &= (M_\tau - M_t) + 2\int_t^\tau e^{\beta s}\delta Y_s \cdot \delta f \mathrm{d}s,\end{aligned}$$

where $M_t = 2\int_0^t e^{\beta s}\delta Y_s \delta Z_s \mathrm{d}W_s$ is a uniformly integrable martingale. Hence

$$\begin{aligned}&\mathbb{E}\left[e^{\beta t}|\delta Y_t|^2\right] + \mathbb{E}\left[\int_t^\tau e^{\beta s}|\delta Z_s|^2 \mathrm{d}s\right]\\ &\leq \mathbb{E}\left[\int_t^\tau e^{\beta s}\left(-\beta|\delta Y_s|^2 + 2C|\delta Y_s|(|y^1 - y^2| + |z^1 - z^2|)\right)\mathrm{d}s\right]\\ &\leq \frac{C^2}{\beta}\mathbb{E}\left[\int_t^\tau e^{\beta s}\left(|y^1 - y^2| + |z^1 - z^2|\right)^2 \mathrm{d}s\right]\\ &\leq \frac{2C^2}{\beta}\mathbb{E}\left[\int_t^\tau e^{\beta s}\left(|y^1 - y^2|^2 + |z^1 - z^2|^2\right)\mathrm{d}s\right].\end{aligned}$$

Therefore, in terms of the β-norms of the solutions:

$$\|\delta Z\|^2_{\mathbb{H}^2_{\beta,\tau}} \leq \frac{2C^2}{\beta}\left\|(y^1 - y^2, z^1 - z^2)\right\|^2_{\mathbb{H}^2_{\beta,\tau}\times\mathbb{H}^2_{\beta,\tau}}.$$

This inequality yields to different estimates for the process δY. The most obvious is that at any time t:

$$e^{\beta t}\mathbb{E}\left[|\delta Y_t|^2\right] \leq \frac{2C^2}{\beta}\left\|(y^1 - y^2, z^1 - z^2)\right\|^2_{\mathbb{H}^2_{\beta,\tau}\times\mathbb{H}^2_{\beta,\tau}}.$$

In particular, since δY_0 is deterministic, we have

$$|\delta Y_0|^2 \leq \frac{2C^2}{\beta}\left\|(y^1 - y^2, z^1 - z^2)\right\|^2_{\mathbb{H}^2_{\beta,\tau}\times\mathbb{H}^2_{\beta,\tau}}.$$

By integrating between 0 and τ both sides of this inequality and remembering that $\tau \leq T$ $\mathbb{P}$-a.s., we obtain:

$$\|\delta Y\|^2_{\mathbb{H}^2_{\beta,\tau}} \leq \frac{2C^2T}{\beta} \left\|(y^1 - y^2, z^1 - z^2)\right\|^2_{\mathbb{H}^2_{\beta,\tau} \times \mathbb{H}^2_{\beta,\tau}} .$$

Therefore, there exists $K > 0$ depending only on C such that

$$\|(\delta Y, \delta Z)\|^2_{\mathbb{H}^2_{\beta,\tau} \times \mathbb{H}^2_{\beta,\tau}} \leq \frac{K}{\beta} \left\|(y^1 - y^2, z^1 - z^2)\right\|^2_{\mathbb{H}^2_{\beta,\tau} \times \mathbb{H}^2_{\beta,\tau}} .$$

Thus, for any $\beta > K$, the map is a contraction. Then by the Banach fixed-point theorem, there exists a unique fixed point $(\overline{Y}, \overline{Z}) \in \mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ which solves the BSDE (rBSDE). □

The hypothesis of Lipschitz continuity can be further weakened. Instead we require f being monotone and Lispchitz in z. This result is shown in Darling and Pardoux (1997).

Theorem 2.3. Let (f, ξ, τ) satisfy

1. *Monotonicity*:

$$(y - \widetilde{y})[f(s, y, z) - f(s, \widetilde{y}, z)] \leq -\alpha |y - \widetilde{y}|^2, \quad \forall y, \widetilde{y} \in \mathbb{R}^n, z \in \mathbb{R}^{n\times d}, \quad \mathbb{P}\text{-a.s.}$$

2. *Lipschitz in z*:

$$|f(s, y, z) - f(s, y, \widetilde{z})| \leq L\,|z - \widetilde{z}|\,, \quad \forall y \in \mathbb{R}^n, z, \widetilde{z} \in \mathbb{R}^{n\times d}, \quad \mathbb{P}\text{-a.s.}$$

3.

$$|f(s, y, z)| \leq |f(s, 0, z)| + \kappa(|y| + \kappa'), \quad \mathbb{P}\text{-a.s.}$$

where $\kappa \geq 0$, and $\kappa' = 0$ or 1.

4. For some $\rho > \gamma$ constants

$$\mathbb{E}\left[e^{\rho\tau}(|\xi|^2 + \kappa') + \int_0^\tau e^{\rho s}\,|f(s, 0, 0)|^2\,\mathrm{d}s\right] < \infty$$

If $f(\cdot, 0, 0) \in \mathbb{H}^2_{\beta,\tau}(\mathbb{R}^n)$ for all $\beta < \gamma$, then (rBSDE) has a unique solution (Y, Z) in $\mathbb{H}^2_\tau(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$.

3

Pontryagin's Maximum Principle: deterministic final time

Stochastic control problems arise in many disciplines where decision-making under uncertainty is crucial. The main goal is to optimize an expected performance criterion, such as efficiency stability, or utility, or minimizing cost, loss or risk, by adjusting a control policy that influences the evolution of a system.

For example, in finance, the classical example (as stated in Lopez De Prado, 2016) is the optimal investment problem introduced and solved in continuous-time by Merton (1971). In the simplest form of the Merton problem, the agent is trying to maximize expected utility of future wealth by trading a risky asset and a risk-free bank account. The agent's actions affect her wealth, but at the same time the uncertain dynamics in the traded asset modulate the agent's wealth in a stochastic manner. The resulting optimal strategies are tied to the dynamics of the asset and perhaps also to the agent's wealth.

The problem of Stochastic Optimal Control has been widely investigated in the literature (Yong & Zhou, 1999, Fleming & Soner, 2006, Pham, 2009, and Touzi, 2013).

In this chapter we will prove the Pontryagin Maximum Principle using an alternative to the spike variation method presented in Yong and Zhou (1999). First we introduce our setting and the stochastic optimal control problems. Then we assess the well-posedness of the control problem. After this, we establish the Gâteaux differentiability of the control-to-state map and the cost functional. Next, we introduce the adjoint equation and use its solution to derive an explicit expression for the gradient of the cost functional. This leads to a stochastic Pontryagin Maximum Principle.

3.1 Problem Formulation

Throughout this chapter let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space on which a d-dimensional standard Brownian motion $W = (W_t)_{t\in[0,T]}$ is defined and let a fixed finite horizon $T > 0$.

We denote by u the *control process*, representing the action, decision, or policy of the decision-maker. At each time t, the controller has access to the information given by the filtration, which contains all events observable up to that moment. However, future uncertainty prevents the controller from anticipating what will happen afterwards. Formally, let $U \subseteq \mathbb{R}^k$ the convex set of possible control values and define $\mathcal{U}_{ad}$ the set of *admissible controls*, with $p > 2$:

$$\mathcal{U}_{ad} := \{u : \Omega\times[0,T] \to U : (\mathcal{F}_t)_t\text{-progressively measurable and } u \in L^p(\Omega\times[0,T];\mathbb{R}^k)\}$$

In order to handle Gâteaux differentiability in a rigorous setting, it will be convenient to embed the admissible control set into a larger, open subset of a Hilbert space. To this end, define

$$\mathcal{U} := L^p(\Omega \times [0,T]; \mathbb{R}^k),$$

and consider an open set $\mathcal{U}'_{ad} \subseteq \mathcal{U}$ such that $\mathcal{U}_{ad} \subseteq \mathcal{U}'_{ad}$.

Given an admissible control $u \in \mathcal{U}_{ad}$, consider the controlled stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \sigma(t, X_t, u_t)\mathrm{d}W_t, & t \in [0,T] \\ X_0 = x_0 \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where the functions

$$b : \Omega \times [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n, \qquad \sigma : \Omega \times [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$$

denote respectively the drift and diffusion coefficients.

We first need to ensure that the state equation (3.1) admits a unique (strong) solution for every admissible control $u \in \mathcal{U}_{ad}$ in the sense of Definition 1.2.4. The process $X = (X_t)_{t\in[0,T]}$ satisfying (3.1) is called the *controlled process*, as its dynamics are driven by the control u.

Let us denote by

$$\mathcal{X} := L^2\left(\Omega;\ C^0([0,T],\mathbb{R}^n)\right),$$

the natural space where X is expected to belong, endowed with the norm

$$\|X\|_{\mathcal{X}}^2 := \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^2\right].$$

where processes are identified up to indinguishability.

The objective of the controller is to minimize the *cost functional*

$$J : \mathcal{U}_{ad} \times \mathcal{X} \to \mathbb{R}, \tag{3.2}$$

$$J(u,X) := \mathbb{E}\left[\int_0^T f(s,X_s,u_s)\mathrm{d}s + h(X_T)\right], \tag{3.3}$$

where

$$f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R} \qquad h : \mathbb{R}^n \to \mathbb{R}$$

are continuous functions representing the *running* and *terminal* costs, respectively.

The stochastic optimal control problem (under strong formulation) can be stated as follows: given the system (3.1) and the cost functional (3.2) find a control $u \in \mathcal{U}_{ad}$ minimizing $J(u,X)$, where X is the corresponding solution of the controlled SDE.

Formally, we seek

$$\overline{u} \in \mathcal{U}_{ad} \quad \text{such that} \quad J(\overline{u},\overline{X}) = \inf_{u\in\mathcal{U}_{ad}} J(u,X),$$

where $\overline{X}$ denotes a strong solution of (3.1) associated with $u = \overline{u}$.

We are just considering the strong solution of the state equation (3.1) and not the weak one, this framework is called *strong formulation*. However, all the discussions and results are readily adaptable to the weak solution. The reason is that only the necessary conditions for optimality are concerned here; hence an optimal (no matter whether in the strong or weak sense) is given as a starting point, and all the results are valid for this given optimal pair on the probability space it attaches to.

3.1.1 Assumptions

We need a number of assumptions for our results. For clarity and ease of reference, we list all assumptions here. The first assumption is important for the well-posedness of the control problem and will be assumed throughout the whole chapter. It gives strong existence and uniqueness of a solution to the state equation, an a-priori bound on this solution and Lipschitz continuity of the control-to-state-map.

Assumptions ($\mathcal{P}$-1). *The coefficients b and σ satisfy*

1. *b and σ are measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^k)$.*
2. *For every $\omega \in \Omega$, $t \in [0,T]$ the maps $(x,u) \mapsto b(\omega,t,x,u)$ and $(x,u) \mapsto \sigma(\omega,t,x,u)$ are $C^{1,1}$, i.e. they are continuously Fréchet differentiable and the derivatives are Lipschitz. In particular, there exist two constants $K > 0$ and $C > 0$ such that for all $t \in [0,T]$, $x, x' \in \mathbb{R}^n$ and $u, u' \in U$ and $\mathbb{P}$-a.s.*

$$\begin{aligned}
|b(t,x,u) - b(t,x',u')| &\leq K(|x-x'| + |u-u'|) \\
|\sigma(t,x,u) - \sigma(t,x',u')| &\leq K(|x-x'| + |u-u'|) \\
|D_x b(t,x,u) - D_x b(t,x',u')| &\leq C(|x-x'| + |u-u'|) \\
|D_u b(t,x,u) - D_u b(t,x',u')| &\leq C(|x-x'| + |u-u'|) \\
|D_x \sigma(t,x,u) - D_x \sigma(t,x',u')| &\leq C(|x-x'| + |u-u'|) \\
|D_u \sigma(t,x,u) - D_u \sigma(t,x',u')| &\leq C(|x-x'| + |u-u'|).
\end{aligned}$$

The second assumption is used to prove the differentiability of the cost functional and the existence of the solution to the adjoint equation.

Assumptions ($\mathcal{P}$-2). *The pair of functions (f,h) satisfy:*

1. *For every* $u \in \mathbb{R}^k$ *the map* $(t,x) \mapsto f(t,x,u)$ *is measurable with respect to* $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^n)$ *and is* C^1*.*
2. *The map* $x \mapsto h(x)$ *is measurable with respect to* $\mathcal{B}(\mathbb{R}^n)$ *and is* C^1*.*
3. *There exists* $C > 0$ *and* $q \in [1, \frac{p}{2})$ *such that for every* $(x,u) \in \mathbb{R}^n \times U$ *and for all* $t \in [0,T]$

$$|D_x f(t,x,u)| \leq C(1 + |x|^q + |u|^q)$$
$$|D_u f(t,x,u)| \leq C(1 + |x|^q + |u|^q)$$
$$|h(x)| \leq C(1 + |x|^q)$$

This assumption requires the coefficients of the state equation to have polynomial growth and has been inspired by Spille and Stannat (2025). This hypothesis is weaker than the quadratic assumption which standard in literature.

Lemma 3.1. If Assumptions ($\mathcal{P}$-1) hold, then there is a constant $C > 0$ such that for every $(t,x,u) \in [0,T] \times \mathbb{R}^n \times U$, $\mathbb{P}$-a.s.

$$|D_x \sigma(t,x,u)| \leq C$$
$$|D_u \sigma(t,x,u)| \leq C$$

Proof. This is clear, by the Lipschitz continuity of σ and the Fréchet differentiability of σ from Assumption ($\mathcal{P}$-1). □

3.2 Well-Posedness of the State Equation

Before proceeding to the derivation of optimality conditions, we establish the exists of a solution to the state equation and state some properties of the control-to-state map.

First of all, we need to show, that a solution to the state equation (3.1) exists and is unique (in the strong sense) for every control $u \in \mathcal{U}_{ad}$ in the sense of Section 1.2.4.

From now on, we will write X^u for a solution to (3.1) to emphasize the dependence on the control u. The following theorem gives the existence and uniqueness of this solution and is a direct consequence of the hypothesis and Theorem 1.13.

Theorem 3.1. Suppose Assumptions ($\mathcal{P}$-**1**) hold. Then for every control $u \in \mathcal{U}$, equation (3.1) has a unique solution in the sense of Definition 1.2.4 and there exists $C > 0$ such that

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X_s^u|^2\right] \leq C(1 + \mathbb{E}\left[|X_0|^2\right]).$$

Corollary 3.1. If Assumptions ($\mathcal{P}$-**1**) hold, then the *control-to-state map*

$$S : \mathcal{U} \to \mathcal{X},$$
$$u \mapsto S(u) := X^u.$$

is well-defined and there exists a constant C such that for all $u, \widetilde{u} \in \mathcal{U}$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^u - X_t^{\widetilde{u}}|^2\right] \leq C\mathbb{E}\left[\int_0^T |u_t - \widetilde{u}_s|^2 \,\mathrm{d}t\right],$$

i.e. the control-to-state map is Lipschitz continuous from $\mathcal{U}$ to $\mathcal{X}$.

Proof. Firstly, since we have a unique strong solution for every $u \in \mathcal{U}$ by Theorem 3.1, the control-to-state map is well-defined.

To prove the estimate, let $u, \widetilde{u} \in \mathcal{U}$ and set

$$Z := X^u - X^{\widetilde{u}}, \qquad t \in [0, T].$$

By Itô's formula 1.11 we obtain

$$|Z_t|^2 = 2\int_0^t \langle Z_s, b(s, X_s^u, u_s) - b(s, X_s^{\widetilde{u}}, \widetilde{u}_s)\rangle + |\sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)|^2 \mathrm{d}s + M_t, \tag{3.4}$$

where

$$M_t := 2\int_0^t \langle Z_s, \sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)\rangle \mathrm{d}W_s$$

is a local martingale (indeed a true martingale under our square-integrability assumptions). We first take the expectations in (3.4). Since M_t is a martingale with $M_0 = 0$ and the integrability hypotheses hold, $\mathbb{E}[M_t] = 0$. Hence

$$\mathbb{E}\left[|Z_t|^2\right] = \mathbb{E}\left[\int_0^t 2\langle Z_s, b(s, X_s^u, u_s) - b(s, X_s^{\widetilde{u}}, \widetilde{u}_s)\rangle + |\sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)|^2 \mathrm{d}s\right].$$

Using the Lipschitz bounds in Assumptions ($\mathcal{P}$-1) and standard inequalities 1.1.4

$$
\begin{aligned}
2\langle Z_s, b(s, X_s^u, u_s) - b(s, X_s^{\widetilde{u}}, \widetilde{u}_s)\rangle &\le 2|Z_s| \cdot |b(s, X_s^u, u_s) - b(s, X_s^{\widetilde{u}}, u_s)| \\
&\le 2K|Z_s| \cdot (|Z_s| + |u_s - \widetilde{u}_s|) \\
&\le 2K \left(|Z_s|^2 + |Z_s| \cdot |u_s - \widetilde{u}_s|\right) \\
&\le 2K \left(|Z_s|^2 + \frac{1}{2}|Z_s|^2 + \frac{1}{2}|u_s - \widetilde{u}_s|^2\right) \\
&\le 4K \left(|Z_s|^2 + |u_s - \widetilde{u}_s|^2\right).
\end{aligned}
$$

For the diffusion term we have for the same reasoning

$$
|\sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)|^2 \le K^2(|Z_s|^2 + |u_s - \widetilde{u}_s|^2).
$$

Combining these two estimates there exists a constant $C_1 > 0$ (depending only on K) such that for all $s \in [0, T]$,

$$
\begin{aligned}
2\langle Z_s, b(s, X_s^u, u_s) - b(s, X_s^{\widetilde{u}}, \widetilde{u}_s)\rangle &+ |\sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)|^2 \\
&\le C_1 \left(|Z_s|^2 + |u_s - \widetilde{u}_s|^2\right).
\end{aligned}
$$

Using Fubini-Tonelli Theorem 1.1, we obtain the Gronwall-type inequality

$$
\mathbb{E}\left[|Z_t|^2\right] \le C \int_0^t \mathbb{E}\left[|Z_s|^2\right] \mathrm{d}s + C\mathbb{E}\left[\int_0^t |u_s - \widetilde{u}_s|^2 \mathrm{d}s\right].
$$

Define $\phi(t) := \mathbb{E}[|Z_t|^2]$ and $g(t) := \mathbb{E}[|u_s - \widetilde{u}_s|^2]$. Then

$$
\phi(t) \le C_1 \int_0^t \phi(s)\, \mathrm{d}s + C_1 \int_0^t g(s)\, \mathrm{d}s.
$$

By Gronwall's inequality 1.6 we deduce

$$
\phi(t) \le C_1 \int_0^t e^{C_1(t-s)} g(s) \mathrm{d}s = C_1 \int_0^t e^{C_1(t-s)} \mathbb{E}\left[|u_s - \widetilde{u}_s|^2\right] \mathrm{d}s. \tag{3.5}
$$

Integrating (3.5) in time yields:

$$
\int_0^T \mathbb{E}\left[|Z_t|^2\right] \mathrm{d}t \le C_1 \int_0^T \int_0^t e^{C_1(t-s)} \mathbb{E}\left[|u_s - \widetilde{u}_s|^2\right] \mathrm{d}s\mathrm{d}t.
$$

By Fubini's Theorem 1.1 and evaluating the inner exponential integral, there exists $C_2 > 0$ (depending on C_1 and T) such that

$$\int_0^T \mathbb{E}\big[|Z_t|^2\big]\mathrm{d}t \le C_2 \int_0^T \mathbb{E}\big[|u_s - \widetilde{u}_s|^2\big]\mathrm{d}s. \tag{3.6}$$

Now we need the uniform-in-time estimate $\mathbb{E}[\sup_{t\in[0,T]} |Z_t|^2]$, the martingale term M_t in (3.4) must be handled by the Burkholder-Davis-Gundy inequality 1.10. We sketch the argument: take supremum in (3.4), then expectations and use BDG 1.10 to bound

$$\mathbb{E}\left[\sup_{s\in[0,t]} |M_s|\right] \le C_{BDG}\,\mathbb{E}\left[\left(\int_0^t |Z_s|^2 |\sigma(s, X_s^u, u_s) - \sigma(s, X_s^{\widetilde{u}}, \widetilde{u}_s)|^2 \mathrm{d}s\right)^{1/2}\right].$$

Applying Young's inequality 1.4 and the Lipschitz bounds one can estimate by a constant C'

$$\mathbb{E}\left[\sup_{s\in[0,t]} |M_s|\right] \le C'\mathbb{E}\left[\int_0^t |Z_s|^2 \mathrm{d}s + \int_0^t |u_s - \tilde{u}_s|^2 \mathrm{d}s\right]$$

Then combining the results, there exists a constant C_3,

$$\mathbb{E}\left[\sup_{s\in[0,t]} |Z_s|^2\right] \le C_3 \int_0^t \mathbb{E}\big[|Z_s|^2\big]\mathrm{d}s + C_3 \int_0^t \mathbb{E}\big[|u_s - \widetilde{u}_s|^2\big]\mathrm{d}s.$$

Combined with (3.6) and Gronwall's lemma this yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Z_t|^2\right] \le C_4 \int_0^T \mathbb{E}\big[|u_s - \widetilde{u}_s|^2\big]\mathrm{d}s,$$

for some constant $C_4 > 0$ depending only on K, T and the BDG constant. By Fubini's Theorem 1.1 we get the thesis. □

Lastly, we notice that thanks to Corollary 3.1, the reduced cost functional $\widetilde{J} : \mathcal{U} \to \overline{\mathbb{R}}$ is well-defined as

$$\widetilde{J}(u) := J(u, S(u))$$

if $f(\cdot, \cdot, S(u), u) \in L^1(\Omega \times [0,T])$ and $h(S(u_T)) \in L^1_T(\Omega)$, and $\widetilde{J}(u) := +\infty$ otherwise.

3.3 Regularity of the Control-to-State Map

We turn to the analysis of the *control-to-state map* and the regularity properties of the corresponding *reduced cost functional*. Recall that, for each admissible control $u \in \mathcal{U}_{ad}$, we denote by X^u the unique strong solution to the state equation (3.1).

Hence, the problem (CP) can be reformulated as follows: minimize $\widetilde{J}(u)$ over all $u \in \mathcal{U}_{ad}$ satisfying the assumptions ($\mathcal{P}$-**1**).

The advantage of this reduced reformulation is that the optimization involves a single variable, the control u. However, this simplification comes at the cost of making the dependence of $\widetilde{J}$ more implicit and analytically more complicated, since it involves the solution of the state equation (3.1). It also makes $\tilde{J}$ not convex even if so are f and h by assumption, because S is non linear. To overcome this issue, we need to study the regularity of the control-to-state map S and of the reduced cost functional $\widetilde{J}$. In particular, we will show that both S and $\widetilde{J}$ are Gâteaux differentiable. We will use the notation $\theta_t = (X_t, u_t)$. In order to study the differentiability of S and, consequently, of $\widetilde{J}$ using variational arguments in the Hilbert space $\mathcal{U}$, we will consider here the control-to-state map S as a map

$$S : \mathcal{U}'_{ad} \to \mathcal{X}$$

3.3.1 Gâteaux Differentiability of the Control-to-State Map

Proposition 3.1. Let Assumption ($\mathcal{P}$-**1**) hold. Then, for every direction $h \in \mathcal{U}$, the equation

$$\begin{aligned} \mathrm{d}Y_t^h &= \left\{ D_x b(t,\theta_t)[Y_t^h] + D_u b(t,\theta_t)[h] \right\} \mathrm{d}t \\ &\quad + \left\{ D_x \sigma(t,\theta_t)[Y_t^h] + D_u \sigma(t,\theta_t)[h] \right\} \mathrm{d}W_t \\ Y_0 &= 0. \end{aligned} \tag{3.7}$$

admits a unique solution Y^h in the sense of Definition 1.2.4.

Proof. For notational simplicity rewrite the linearized equation (3.7) in the compact form

$$\mathrm{d}Y_t = \widetilde{b}(t, Y_t^h)\mathrm{d}t + \widetilde{\sigma}(t, Y_t^h)\mathrm{d}W_t, \quad Y_0 = 0,$$

where, for a fixed direction h in the control space, we set

$$\widetilde{b}(\omega, t, y) := D_x b(t, \theta_t(\omega))[y] + D_u b(t, \theta_t(\omega))[h],$$
$$\widetilde{\sigma}(\omega, t, y) := D_x \sigma(t, \theta_t(\omega))[y] + D_u \sigma(t, \theta_t(\omega))[h].$$

By construction X^u and u are progressively-measurable processes, hence the mapping $\omega \mapsto \theta_t(\omega)$ is $\mathcal{F}_t$-measurable for each t. Under the standing regularity assumptions the derivatives $D_x b$, $D_u b$, $D_x \sigma$, $D_u \sigma$ are Borel measurable, so the compositions $\omega \mapsto D_x b(t, \theta_t(\omega))$, etc., are $\mathcal{F}_t$-measurable. Therefore for each fixed y the processes $(\omega, t) \mapsto \widetilde{b}(\omega, t, y)$ and $(\omega, t) \mapsto \widetilde{\sigma}(\omega, t, y)$ are progressively measurable. At this point since by assumptions b and σ are Lipschitz and their derivatives are bounded, we have that $\widetilde{b}$ and $\widetilde{\sigma}$ satisfy the standard assumptions for existence and uniqueness of the solution to an SDE, Theorem 1.13. Hence, there exists a unique solution $Y^h \in \mathcal{X}$ to (3.7). □

They key step now is to establish the Gâteaux differentiability of the control-to-state map.

Lemma 3.2. Under Assumptions ($\mathcal{P}$-**1**), for every direction $h \in \mathcal{U}$, the solution Y^h of the linearized equation (3.7) satisfies the estimate

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_t^h\right|^2\right] \leq C\|h\|_{\mathcal{U}}^2,$$

for some constant $C > 0$ independent of h. Consequently, the map $h \mapsto Y^h$ is a bounded linear operator $\mathcal{U} \to \mathcal{X}$.

Proof. The proof follows from standard estimates for stochastic differential equations with Lipschitz coefficients. Applying Itô's formula 1.11 to $|Y_t^h|^2$ (using a C^1

approximation of the norm if necessary) yields

$$\begin{aligned}\left|Y_t^h\right|^2 = &\int_0^t 2\left\langle Y_s^h, D_x b(s,\theta_s)[Y_s^h] + D_u b(s,\theta_s)[h]\right\rangle \mathrm{d}s \\ &+ \int_0^t \left|D_x\sigma(s,\theta_s)[Y_s^h] + D_u\sigma(s,\theta_s)[h]\right|^2 \mathrm{d}s \\ &+ 2\int_0^t \left\langle Y_s^h, D_x\sigma(s,\theta_s)[Y_s^h] + D_u\sigma(\theta_s)[h]\right\rangle \mathrm{d}W_s,\end{aligned}$$

where the last integral defines a local martingale, denoted by

$$M_t := 2\int_0^t \left\langle Y_s^h, D_x\sigma(s,\theta_s)[Y_s^h] + D_u\sigma(s,\theta_s)[h]\right\rangle \mathrm{d}W_s.$$

By boundedness of the derivatives of b and σ as per ($\mathcal{P}$-1), there exists a constant $C > 0$ such that

$$\left\langle Y_s^h, D_x b(s,\theta_s)[Y_s^h]\right\rangle \leq C|Y_s^h|^2.$$

Using Young's inequality 1.4, for every $\varepsilon > 0$,

$$\left\langle Y_s^h, D_u b(s,\theta_s)[h]\right\rangle \leq \frac{1}{\varepsilon}|Y_s^h|^2 + \varepsilon\underbrace{|D_u b(s,\theta_s)[h]|^2}_{\leq C|h|^2},$$

and similarly

$$|D_u\sigma(s,\theta_s)[h]|^2 \leq C|h|^2.$$

Combining the above estimates gives

$$|Y_t^h|^2 \leq C\int_0^t |Y_s^h|^2 \mathrm{d}s + C\int_0^t |h_s|^2 \mathrm{d}s + M_t.$$

Taking the supremum in time and the expectation, and applying Burkholder-Davis-Gundy inequality 1.10 and Young's inequality 1.4, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^h|^2\right] &\leq C\mathbb{E}\left[\left(\int_0^T |Y_s^h|^2\left|D_x b(s,\theta_s)[Y_s^h] + D_u b(s,\theta_s)[h]\right|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^h|^2\right] + C\mathbb{E}\left[\int_0^T |Y_s^h|^2 + |h_s|^2 \mathrm{d}s\right].\end{aligned}$$

Absorbing the first term into the left-hand side gives

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^h|^2\right] \leq C\int_0^T \mathbb{E}\left[\sup_{r\in[0,s]} |Y_r^h|^2\right] \mathrm{d}s + C|h|^2.$$

Define $\phi(t) := \mathbb{E}\big[\sup_{r\in[0,t]} |Y_r^h|^2\big]$. Then the above inequality implies

$$\phi(t) \leq C\int_0^t \phi(s)\,\mathrm{d}s + C\,|h|^2.$$

By Gronwall's lemma,

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^h|^2\right] \leq C\,|h|^2,$$

as claimed. This concludes the proof of the intermediate result. □

We are now ready to assess the Gâteaux differentiability of the control-to-state function.

Theorem 3.2. If Assumptions ($\mathcal{P}$-**1**) hold, the control-to-state map

$$S : \mathcal{U}'_{ad} \to \mathcal{X}$$

is Gâteaux differentiable. Moreover, for each $u \in \mathcal{U}'_{ad}$, its derivative in the direction $h \in \mathcal{U}$ is given by

$$dS(u)[h] = Y^h.$$

Proof. Let $\epsilon > 0$ such that $u + \epsilon h \in \mathcal{U}'_{ad}$.

Denote by X the solution of the state equation (3.1) associated with the control u, and by X^ϵ the solution corresponding to the (3.1) with control $u + \epsilon h$.

For each $\lambda \in [0, 1]$, define the convex interpolation between X and X^ϵ

$$X^{\lambda,\epsilon} := X + \lambda(X^\epsilon - X), \qquad u^{\lambda,\epsilon} := u + \lambda\epsilon h,$$

and use the notation

$$\theta_t = (X_t, u_t), \quad \theta_t^\epsilon = (X_t^\epsilon, u_t + \epsilon h), \quad \theta_t^{\lambda,\epsilon} = (X_t^{\lambda,\epsilon}, u^{\lambda,\epsilon}).$$

By the Fréchet differentiability of $b(t,\cdot,\cdot)$ and $\sigma(t,\cdot,\cdot)$ as in Assumption ($\mathcal{P}$-1), we can expand by Taylor for each $t \in [0,T]$

$$b(t,\theta_t^\epsilon) - b(t,\theta_t) = \int_0^1 \Big(D_x b(t,\theta_t^{\lambda,\epsilon})[X_t^\epsilon - X_t] + D_u b(t,\theta_t^{\lambda,\epsilon})[\epsilon h] \Big)\, \mathrm{d}\lambda,$$

and similarly,

$$\sigma(t,\theta_t^\epsilon) - \sigma(t,\theta_t) = \int_0^1 \Big(D_x \sigma(t,\theta_t^{\lambda,\epsilon})[X_t^\epsilon - X_t] + D_u \sigma(t,\theta_t^{\lambda,\epsilon})[\epsilon h] \Big)\, \mathrm{d}\lambda.$$

We define

$$\Delta^\epsilon := \frac{X^\epsilon - X}{\epsilon} - Y^h,$$

and aim to show that $\|\Delta^\epsilon\|_{\mathcal{X}} \to 0$ or, equivalently

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta_t^\epsilon|^2\right] \to 0, \quad \text{as} \quad \epsilon \to 0.$$

Using Itô's Lemma 1.11 and substituting the expression for Y^h we get:

$$\begin{aligned}
\mathrm{d}\Delta_t^\epsilon = & \Bigg[\frac{b(t,\theta_t) + \int_0^1 \big(D_x b(s,\theta_s^{\lambda,\epsilon})[X_s^\epsilon - X_s] + \epsilon D_u b(s,\theta_s^{\lambda,\epsilon})[h] \big)\, \mathrm{d}\lambda - b(t,\theta_t)}{\epsilon} \\
& \quad - D_x b(t,\theta_t)[Y_t^h] - D_u b(t,\theta_t)[h] \Bigg] \mathrm{d}t \\
& \Bigg[\frac{\sigma(t,\theta_t) + \int_0^1 \big(D_x \sigma(s,\theta_s^{\lambda,\epsilon})[X_s^\epsilon - X_s] + \epsilon D_u \sigma(s,\theta_s^{\lambda,\epsilon})[h] \big)\, \mathrm{d}\lambda - \sigma(t,\theta_t)}{\epsilon} \\
& \quad - D_x \sigma(t,\theta_t)[Y_t^h] - D_u \sigma(t,\theta_t)[h] \Bigg] \mathrm{d}W_t.
\end{aligned}$$

Using the integral form and noting that all derivatives involved are Fréchet (hence they are linear in the direction), one obtains:

$$\begin{aligned}
\Delta_t^\epsilon = & \int_0^t \underbrace{\int_0^1 D_x b(s,\theta_s^{\lambda,\epsilon})[\Delta_s^\epsilon]\mathrm{d}\lambda}_{I_1(s)}\,\mathrm{d}s + \int_0^t \underbrace{\int_0^1 D_x b(s,\theta_s^{\lambda,\epsilon})[Y_s^h] - D_x b(\theta_s)[Y_s^h]\mathrm{d}\lambda}_{I_2(s)}\,\mathrm{d}s \\
& + \int_0^t \underbrace{\int_0^1 D_u b(s,\theta_s^{\lambda,\epsilon})[h] - D_u b(s,\theta_s)[h]\mathrm{d}\lambda}_{I_3(s)}\,\mathrm{d}s \\
& + \int_0^t \underbrace{\int_0^1 D_x \sigma(s,\theta_s^{\lambda,\epsilon})[\Delta_s^\epsilon]\mathrm{d}\lambda}_{I_4(s)}\,\mathrm{d}W_s + \int_0^t \underbrace{\int_0^1 D_x \sigma(s,\theta_s^{\lambda,\epsilon})[Y_s^h] - D_x \sigma(s,\theta_s)[Y_s^h]\mathrm{d}\lambda}_{I_5(s)}\,\mathrm{d}W_s \\
& + \int_0^t \underbrace{\int_0^1 D_u \sigma(s,\theta_s^{\lambda,\epsilon})[h] - D_u \sigma(s,\theta_s)[h]\mathrm{d}\lambda}_{I_6(s)}\,\mathrm{d}W_s \\
= & \int_0^t (I_1(s) + I_2(s) + I_3(s))\,\mathrm{d}s + \int_0^t (I_4(s) + I_5(s) + I_6(s))\,\mathrm{d}W_s.
\end{aligned}$$

This separation allows us to show that each I_i is addressing a different direction. Now use Itô's Lemma 1.11 and we get:

$$\begin{aligned}
\mathrm{d}|\Delta_t^\epsilon|^2 = & \Big\{ 2\langle \Delta_t^\epsilon, I_1(t) + I_2(t) + I_3(t)\rangle + |I_4(t) + I_5(t) + I_6(t)|^2 \Big\}\mathrm{d}t \\
& + 2\langle \Delta_t^\epsilon, I_4(t) + I_5(t) + I_6(t)\rangle \mathrm{d}W_t.
\end{aligned}$$

We can rewrite this as:

$$\begin{aligned}
|\Delta_t^\epsilon|^2 = & \int_0^t 2\langle \Delta_s^\epsilon, I_1(s)\rangle \mathrm{d}s + \int_0^t 2\langle \Delta_s^\epsilon, I_2(s) + I_3(s)\rangle \mathrm{d}s \\
& + \int_0^t |I_4(s)|^2 + |I_5(s) + I_6(s)|^2 + 2\langle I_4(s), I_5(s) + I_6(s)\rangle \mathrm{d}s \\
& + \int_0^t 2\langle \Delta_s^\epsilon, I_4(s) + I_5(s) + I_6(s)\rangle \mathrm{d}W_s.
\end{aligned}$$

Now we can estimate:

$$\begin{aligned}\|\Delta^\epsilon\|_{\mathcal{X}}^2 \leq \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t 2\langle\Delta_s^\epsilon, I_1(s)\rangle + |I_4(s)|^2 \mathrm{d}s + \int_0^t 2\langle\Delta_s^\epsilon, I_2(s)+I_3(s)\rangle \mathrm{d}s\right] \\ + \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t |I_5(s)+I_6(s)|^2 + 2\langle I_4(s), I_5(s)+I_6(s)\rangle \mathrm{d}s\right] \\ + \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t 2\langle\Delta_s^\epsilon, I_4(s)+I_5(s)+I_6(s)\rangle \mathrm{d}W_s\right].\end{aligned}$$

Now let's estimate each term separately using Young's inequality 1.4:

$$\int_0^t 2\langle\Delta_s^\epsilon, I_2(s)+I_3(s)\rangle \mathrm{d}s \leq \int_0^t |\Delta_s^\epsilon|^2 + |I_2(s)+I_3(s)|^2 \mathrm{d}s.$$

So it can be bounded as:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t |\Delta_s^\epsilon|^2 + |I_2(s)+I_3(s)|^2 \mathrm{d}s\right] \\ \leq \mathbb{E}\left[\int_0^T |\Delta_s^\epsilon|^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T |I_2(s)+I_3(s)|^2 \mathrm{d}s\right].\end{aligned}$$

By the same reasoning we can estimate the row as:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t |I_5(s)+I_6(s)|^2 + 2\langle I_4(s), I_5(s)+I_6(s)\rangle \mathrm{d}s\right] \\ \leq \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t |I_4(s)|^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^T |I_5(s)+I_6(s)|^2 \mathrm{d}s\right].\end{aligned}$$

For the third row we use the Burkholder-Davis-Gundy inequality 1.10 to get (since the the stochastic integral is a local martingale):

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t 2\langle\Delta_s^\epsilon, I_4(s)+I_5(s)+I_6(s)\rangle \mathrm{d}W_s\right] \\ \leq C\mathbb{E}\left[\left(\int_0^t |\Delta_t^\epsilon|^2 \cdot |I_4(s)+I_5(s)+I_6(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right].\end{aligned}$$

So putting things together:

$$\|\Delta^\epsilon\|_{\mathcal{X}}^2 \leq \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t 2\langle \Delta_s^\epsilon, I_1(s)\rangle + 2|I_4(s)|^2 \mathrm{d}s + \int_0^T |\Delta_s^\epsilon|^2 \mathrm{d}s\right] \tag{A}$$

$$+ 2\mathbb{E}\left[\int_0^T |I_2(s)+I_3(s)|^2 + |I_5(s)+I_6(s)|^2 \mathrm{d}s\right] \tag{B}$$

$$+ C\mathbb{E}\left[\left(\int_0^t |\Delta_t^\epsilon|^2 \cdot |I_4(s)+I_5(s)+I_6(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right]. \tag{C}$$

Again we split the analysis by the 3 rows of this equation. The first one can be estimated noting that by Assumption ($\mathcal{P}$-1) we have that for every $z \in \mathbb{R}^n$ and every $t \in [0,T]$ we have that $\langle D_x b(t,\theta_t)[z], z\rangle \leq C'|z|^2$.

$$\begin{aligned}
(A) &\leq \mathbb{E}\Bigg[\sup_{t\in[0,T]}\int_0^t 2\langle \Delta_s^\epsilon, \int_0^1 D_x b(s,\theta_s^{\lambda,\epsilon})[\Delta_s^\epsilon]\rangle \mathrm{d}s \\
&\quad + 2\int_0^t \Big|\int_0^1 \underbrace{D_x\sigma(s,\theta_s^{\lambda,\epsilon})[\Delta_s^\epsilon]}_{\leq C'|\Delta_s^\epsilon|^2} \mathrm{d}\lambda\Big|^2 \mathrm{d}s + \int_0^T |\Delta_s^\epsilon|^2 \mathrm{d}s\Bigg] \\
&\leq C_2 \int_0^T \mathbb{E}\left[\sup_{s\in[0,t]} |\Delta_s^\epsilon|^2 \mathrm{d}t\right].
\end{aligned}$$

By Young inequality 1.4 and by following the same reasoning as in (A) we get:

$$\begin{aligned}
(C) &\leq \mathbb{E}\left[\left(\sup_{t\in[0,T]} |\Delta_t^\epsilon|^2 \cdot 4\int_0^T |I_4(s)+I_5(s)+I_6(s)|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta_t^\epsilon|^2\right] + 2\mathbb{E}\left[\int_0^T |I_4(s)+I_5(s)+I_6(s)|^2 \mathrm{d}s\right] \\
&\leq \frac{1}{2}C_3 \|\Delta^\epsilon\|_{\mathcal{X}}^2 + 4\mathbb{E}\left[\int_0^T |I_5(s)+I_6(s)|^2 \mathrm{d}s\right].
\end{aligned}$$

So putting things together we have:

$$
\begin{aligned}
\|\Delta^\epsilon\|_{\mathcal{X}} \leq & \int_0^T C_4 \mathbb{E}\left[\sup_{s\in[0,t]} |\Delta_s^\epsilon|^2\right] \mathrm{d}t + \\
& + 12\mathbb{E}\left[\int_0^T |I_2(s)+I_3(s)|^2 + |I_5(s)+I_6(s)|^2 \mathrm{d}s\right] \\
\leq & \, C_5 \mathbb{E}\left[\int_0^T |I_2(s)|^2 + |I_3(s)|^2 + |I_5(s)|^2 + |I_6(s)|^2 \mathrm{d}s\right].
\end{aligned}
$$

It is left to show that the right-hand side converges to 0 as $\epsilon \to 0$. We show it for I_2 only, the others are similar. By the Lipschitz continuity of $D_x b$ we have:

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T |I_2(s)|^2 \mathrm{d}s\right] & \leq \int_0^1 \mathbb{E}\left[\int_0^T |(D_x\sigma(s,\theta_s^{\lambda,\epsilon}) - D_x\sigma(s,\theta_s))[Y_s^h]|^2 \mathrm{d}s\right] \mathrm{d}\lambda \\
& \leq C\int_0^1 \mathbb{E}\left[\int_0^T M(1+|X_s^{\lambda,\epsilon}-X_s|^2 + |\lambda\epsilon h|^2)\mathrm{d}s\right]\mathrm{d}\lambda \\
& \leq C\underbrace{\int_0^1 \mathbb{E}\left[\int_0^T \lambda^2\epsilon^2 M|h|^2 \mathrm{d}s\right]\mathrm{d}\lambda}_{\leq C\epsilon^2|h|^2\to 0} + C'\underbrace{\int_0^1 \mathbb{E}\left[\int_0^T |X_s^{\lambda,\epsilon}-X_s|^2\mathrm{d}s\right]\mathrm{d}\lambda}_{\leq C\sup_{\lambda\in[0,1]}\mathbb{E}\left[\int_0^T |X^{\lambda,\epsilon}-X|^2\mathrm{d}s\right]}.
\end{aligned}
$$

So we only need to show that, for every λ and $\epsilon \to 0$

$$
\mathbb{E}\left[\int_0^T |X_s^{\lambda,\epsilon} - X_s|^2 \mathrm{d}s\right] \to 0
$$

But this is guaranteed by the Lemma 3.1, and so, by iterating this procedure for also I_3, I_4, I_5 we get

$$
\mathbb{E}\left[\sup_{t\in[0,T]} |\Delta_t^\epsilon|^2\right] \to 0 \quad \Leftrightarrow \quad \|\Delta^\epsilon\|_{\mathcal{X}} \to 0,
$$

and the control to state map is Gâteaux differentiable. □

Note that the Gâteaux derivative of the control-to-state operator is linear in the direction, that is,

$$
DS(u)[\cdot] \in \mathcal{L}(\mathcal{U};\mathcal{X}).
$$

The uniqueness of the solution to the linearized SDE (3.7) allows us to verify linearity by checking whether the process $Y^{\alpha h_1+\beta h_2}$ satisfies the same equation as $\alpha Y^{h_1} + \beta Y^{h_2}$

for any $\alpha, \beta \in \mathbb{R}$ and $h_1, h_2 \in \mathcal{U}$.

Indeed, by definition of Y^h, we have:

$$\begin{aligned}
\mathrm{d}(Y^{h_1+h_2}) &= \left\{D_x b(t,\theta_t)[Y^{h_1+h_2}] + D_u b(t,\theta_t)[h_1+h_2]\right\} \mathrm{d}t \\
&\quad + \left\{D_x \sigma(t,\theta_t)[Y^{h_1+h_2}] + D_u \sigma(\theta)[h_1+h_2]\right\} \mathrm{d}W_t \\
\mathrm{d}(Y^{h_1} + Y^{h_2}) &= \left\{D_x b(t,\theta_t)[Y^{h_1} + Y^{h_2}] + D_u b(t,\theta_t)[h_1+h_2]\right\} \mathrm{d}t \\
&\quad + \left\{D_x \sigma(t,\theta_t)[Y^{h_1} + Y^{h_2}] + D_u \sigma(t,\theta_t)[h_1+h_2]\right\} \mathrm{d}W_t
\end{aligned}$$

Since both b and σ are continuously Fréchet differentiable, their derivatives are linear maps, and therefore these two SDEs coincide. By uniqueness of solutions, it follows that

$$Y^{\alpha h_1 + \beta h_2} = \alpha Y^{h_1} + \beta Y^{h_2},$$

which shows that $DS(u)$ is a linear operator from $\mathcal{U}_{ad}$ into $\mathcal{X}$.

3.3.2 Fréchet Differentiability of the Control-to-State Map

Since Gâteaux differentials do not permit to apply chain rule and the reduced cost functional is a composed map, we need the Fréchet differentiability of the control-to-state map.

Theorem 3.3. If Assumptions ($\mathcal{P}$-1) hold, then the control-to-state map is Fréchet differentiable.

Proof. We want to verify the hypothesis of Theorem 1.2. We need to prove that the mapping

$$u \mapsto DS(u) \in \mathcal{L}(\mathcal{U}, \mathcal{X}),$$

is continuous.

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}$ be a sequence converging to $u \in \mathcal{U}$. For each n denote by $Y^{n,h}$ the solution of the linearized equation (3.7) with control u_n, and by Y^h the one corresponding to u. We need to show that

$$\sup_{\substack{h\in\mathcal{U} \\ \|h\|_{\mathcal{U}}=1}} \left\|Y^{n,h} - Y^h\right\|_{\mathcal{X}} \to 0.$$

Define $\delta^{n,h} := Y^{n,h} - Y^h$. Using, again the notation $\theta_t^n = (X_t^{u_n}, u_n)$, and subtracting the two SDEs for $Y^{n,h}$ and Y^h, we obtain

$$\begin{aligned}
\mathrm{d}\delta_t^{n,h} &= \Big\{ D_x b(t,\theta_t^n)[Y_t^{n,h}] - D_x b(t,\theta_t)[Y_t^h] + D_u b(t,\theta_t^n)[h] - D_u b(t,\theta_t)[h] \Big\}\, \mathrm{d}t \\
&\quad + \Big\{ D_x \sigma(t,\theta_t^n)[Y_t^{n,h}] - D_x \sigma(t,\theta_t)[Y_t^h] + D_u \sigma(t,\theta_t^n)[h] - D_u \sigma(t,\theta_t)[h] \Big\}\, \mathrm{d}W_t.
\end{aligned}$$

Rearranging the terms, we can write

$$\begin{aligned}
\mathrm{d}\delta_t^{n,h} = {} & D_x b(t,\theta_t^n)[\delta_t^{n,h}] \\
& + \underbrace{(D_x b(t,\theta_t^n) - D_x b(t,\theta_t))[Y_t^h] + (D_u b(t,\theta_t^n) - D_u b(t,\theta_t))[h]}_{r_t^{n,h}}\, \mathrm{d}t \\
& + D_x \sigma(t,\theta_t^n)[\delta_t^{n,h}] \\
& + \underbrace{(D_x \sigma(t,\theta_t^n) - D_x \sigma(t,\theta_t))[Y_t^h] + (D_u \sigma(t,\theta_t^n) - D_u \sigma(t,\theta_t))[h]}_{s_t^{n,h}}\, \mathrm{d}W_t.
\end{aligned}$$

Applying Itô's formula 1.11 to $|\delta_t|^2$, we obtain:

$$\begin{aligned}
|\delta_t^{n,h}|^2 = & \int_0^t 2\langle \delta_s^{n,h}, D_x b(s,\theta_s^n)[\delta_s^{n,h}] + r_s^{n,h} \rangle + |D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h}|^2 \mathrm{d}s \\
& + \int_0^t 2\langle \delta_s^{n,h}, D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h} \rangle \mathrm{d}W_s.
\end{aligned}$$

Using the Lipschitz continuity and boundedness of the derivatives of b and σ, together with Cauchy–Schwarz 1.5 and Young 1.4 inequalities, there exists a constant $C > 0$ such that

$$\begin{aligned}
|D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h}|^2 &= |D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}]|^2 + |s_s^{n,h}|^2 + 2\langle D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}], s_s^{n,h} \rangle \\
&\leq C(|\delta_s^{n,h}|^2 + |s_s^{n,h}|^2), \\
2\langle \delta_s^{n,h}, D_x b(s,\theta_s^n)[\delta_s^{n,h}] + r_s^{n,h} \rangle &\leq C(|\delta_s^{n,h}|^2 + |r_s^{n,h}|^2).
\end{aligned}$$

Hence,

$$|\delta_t^{n,h}|^2 \leq C \int_0^t |\delta_s^{n,h}|^2 + |r_s^{n,h}|^2 + |s_s^{n,h}|^2 \mathrm{d}s + M_t,$$

where M_t

$$M_t := \int_0^t 2\langle \delta_s^{n,h}, D_x \sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h} \rangle \mathrm{d}W_s$$

is a local martingale. Taking the supremum in time and the expectation:

$$\left\|\delta^{n,h}\right\|_{\mathcal{X}}^2 \leq C\mathbb{E}\left[\int_0^T |\delta_s^{n,h}|^2 + |r_s^{n,h}|^2 + |s_s^{n,h}|^2 \mathrm{d}s\right] + \mathbb{E}\left[\sup_{t\in[0,T]} M_t\right].$$

Using the Burkholder–Davis–Gundy 1.10, Cauchy–Schwarz 1.5, and Young 1.4 inequalities , we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} M_t\right] &\leq C\mathbb{E}\left[\left(\int_0^T 4|\delta_s^{n,h}|^2 \cdot |D_x\sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h}|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq C\mathbb{E}\left[\left(\sup_{s\in[0,T]} |\delta_s^{n,h}|^2 \cdot \int_0^T |D_x\sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^n|^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq C\mathbb{E}\left[\sup_{s\in[0,T]} |\delta_s^{n,h}|^2\right] + C\mathbb{E}\left[\int_0^T |D_x\sigma(s,\theta_s^n)[\delta_s^{n,h}] + s_s^{n,h}|^2 \mathrm{d}s\right] \\
&\leq \frac{1}{2}\left\|\delta^{n,h}\right\|_{\mathcal{X}}^2 + C\mathbb{E}\left[\int_0^T |\delta_s^{n,h}|^2 + |s_s^{n,h}|^2 \mathrm{d}s\right].
\end{aligned}$$

Combining the estimates yields

$$\left\|\delta^{n,h}\right\|_{\mathcal{X}}^2 \leq C\int_0^T \left|\delta_s^{n,h}\right|^2 \mathrm{d}s + C\mathbb{E}\left[\int_0^T |r_s^{n,h}|^2 + |s_s^{n,h}|^2 \mathrm{d}s\right].$$

Taking the supremum over all h such that $\|h\|_{\mathcal{U}} \leq 1$, we conclude that

$$\begin{aligned}
\sup_{\|h\|_{\mathcal{U}}\leq 1} \|Y^{n,h} - Y^h\|_{\mathcal{X}}^2 \leq C\int_0^T \sup_{\|h\|_{\mathcal{U}}\leq 1} \|Y^{n,h} - Y^h\|_{\mathcal{X}}^2 \, \mathrm{d}s \\
+ C \sup_{\|h\|_{\mathcal{U}}\leq 1} \mathbb{E}\left[\int_0^T (|r_s^{n,h}|^2 + |s_s^{n,h}|^2)\mathrm{d}s\right].
\end{aligned}$$

By Lipschitzianity and sublinear growth of $D_x b$ and $D_u b$ we have that:

$$|r_t^{n,h}| \leq L(|X_t^{u_n} - X_t^u| + |u_{n,t} - u_t|)(|Y_t^h| + |h|)$$

Using Young's inequality we get

$$\mathbb{E}\left[\int_0^T |r^{n,h}|^2 \mathrm{d}t\right] \leq C\mathbb{E}\left[\int_0^T |X_t^{u_n} - X_t^u|^2 + |u_{n,t} - u_t|^2 \mathrm{d}t\right] \cdot \left(\sup_{\|h\|_{\mathcal{U}}\leq 1} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^h|^2 + 1\right]\right).$$

From Lemma 3.2 we know that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^h|^2\right] \leq C\|h\|_{\mathcal{U}}^2.$$

Then

$$\mathbb{E}\left[\int_0^T |r_t^{n,h}|^2 \mathrm{d}t\right] \leq C\mathbb{E}\left[\int_0^T (|X_t^{u_n} - X_t^u|^2 + |u_{n,t} - u_t|^2)\mathrm{d}t\right] \to 0, \quad \text{as } n \to \infty.$$

This holds uniformly in h such that $\|h\|_{\mathcal{U}} \leq 1$. The same argument applies to $s^{n,h}$. This implies

$$\sup_{\|h\|_{\mathcal{U}}\leq 1} \|Y^{n,h} - Y^h\|_{\mathcal{X}} \to 0.$$

Therefore, the control-to-state map S is continuously Fréchet differentiable from $\mathcal{U}'_{ad}$ into $\mathcal{X}$ for 1.2. □

3.3.3 Gâteaux differentiability of the Cost Functional

An application of the chain rule gives the Gâteaux differentiability of the cost functional.

Proposition 3.2. Let Assumptions ($\mathcal{P}$-1) and ($\mathcal{P}$-2) hold. Then the reduced cost functional $\widetilde{J}$ is Gâteaux differentiable at every point $u \in \mathcal{U}'_{ad}$ and its Gâteaux derivative in the direction $h \in \mathcal{U}$ is given by

$$D\widetilde{J}(u)[h] = \mathbb{E}\left[\int_0^T D_x f(t, X^u, u)[Y_t^h] + D_u f(t, X^u, u)[h]\mathrm{d}t + D_x h(X_T^u)[Y_T^h]\right].$$

Proof. The proof follows the same reasoning as that used for the Gâteaux differentiability of the control-to-state map. Fix $u \in \mathcal{U}'_{ad}$, $h \in \mathcal{U}$ and let $\epsilon > 0$ be such that

$u + \epsilon h \in \mathcal{U}'_{ad}$. Then, by definition of $\widetilde{J}$,

$$\frac{\widetilde{J}(u+\epsilon h)-\widetilde{J}(u)}{\epsilon} = \mathbb{E}\left[\int_0^T \frac{f(t,\theta_t^\epsilon)-f(t,\theta_t)}{\epsilon}\mathrm{d}t + \frac{h(X_t^\epsilon)-h(X_T)}{\epsilon}\right],$$

where, as before, we set $\theta_t = (X_t, u_t)$ and $\theta_t^\epsilon = (X_t^\epsilon, u_t + \epsilon h)$.

Since f and h are continuously Fréchet differentiable by assumption, we can write their Taylor expansions in the integral form for all $t \in [0,T]$ with the same notation as before.

$$f(t,\theta_t^\epsilon) - f(t,\theta_t) = \int_0^1 \big(D_x f(t,\theta_t^{\lambda,\epsilon})[X_t^\epsilon - X_t] + D_u f(t,\theta_t^{\lambda,\epsilon})[\epsilon h]\big)\mathrm{d}\lambda,$$
$$h(X_T^\epsilon) - h(X_T) = \int_0^1 D_x h(X_T^{\lambda,\epsilon})[X_T^\epsilon - X_T]\mathrm{d}\lambda,$$

where $\theta_t^{\lambda,\epsilon} := (X_t^{\lambda,\epsilon}, u_t + \lambda\epsilon h)$ and $X^{\lambda,\epsilon} := X + \lambda(X^\epsilon - X)$.

Dividing by ϵ and exploiting the linearity of the Fréchet derivatives, we obtain

$$\frac{f(t,\theta_t^\epsilon)-f(t,\theta_t)}{\epsilon} = \int_0^1 D_x f(t,\theta_t^{\lambda,\epsilon})\left[\frac{X_t^\epsilon - X_t}{\epsilon}\right] + D_u f(t,\theta_t^{\lambda,\epsilon})[h]\mathrm{d}\lambda.$$

From the proof of the Gâteaux differentiability of the control-to-state map, we know that

$$\frac{X_t^\epsilon - X_t}{\epsilon} \to Y_t^h \quad \text{in } \mathcal{X}, \text{ as } \epsilon \to 0.$$

To justify the passage of the limit under the integrals, we verify that the integrands are uniformly dominated. By the existence and uniqueness result for strong solutions of SDEs 3.2 and 1.13, there exists $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^\epsilon|^2\right], \quad \mathbb{E}\left[\sup_{t\in[0,T]} \left|\frac{X_t^\epsilon - X_t}{\epsilon}\right|^2\right], \quad \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^h|^2\right] \le C(1+|u|^2+|h|^2).$$

Moreover, from the growth assumptions on f and its derivatives, there exists $C > 0$ and $q \ge 0$ such that:

$$|D_x f(t,x,u)| \le C(1+|x|^q+|u|^q),$$

and analogous estimates hold for $D_u f$ and $D_x h$. Indeed, we have $X \in L^p(\Omega; C^0([0,T];\mathbb{R}^n))$ because $u \in L^p(\Omega; L^p(0,T;\mathbb{R}^n))$. We know that the incremental ratio $\frac{X_t^\epsilon - X_t}{\epsilon}$ converges in L^2, so it is sufficient that $D_x f$ is uniformly dominated in L^r with $r > 2$. This is true by taking $r = \frac{p}{q} > 2$ by Assumption ($\mathcal{P}$-2).

Hence, the integrands are dominated by an integrable function independent of ϵ. By continuity of the Fréchet derivative and convergence of $\frac{X_t^\epsilon - X_t}{\epsilon}$ to Y_t^h, we have in $L^1(\Omega \times [0,T])$:

$$\begin{aligned} D_x f(t, \theta_t^{\lambda,\epsilon}) \left[\frac{X_t^\epsilon - X_t}{\epsilon} \right] &\to D_x f(t, \theta_t)[Y_t^h], \\ D_u f(t, \theta_t^{\lambda,\epsilon})[h] &\to D_u f(t, \theta_t)[h], \\ D_x h(X_T^{\lambda,\epsilon}) \left[\frac{X_T^\epsilon - X_T}{\epsilon} \right] &\to D_x h(X_T)[Y_T^h]. \end{aligned}$$

By the Dominated Convergence Theorem, we can exchange limit and expectation, obtaining the desired result:

$$D\widetilde{J}(u)[h] = \mathbb{E}\left[\int_0^T D_x f(t, X_t^u, u_t)[Y_t^h] + D_u f(t, X_t^u, u_t)[h] \, \mathrm{d}t + D_x h(X_T^u)[Y_T^h] \right].$$

This completes the proof. □

3.4 Adjoint Equation and the Pontryagin Maximum Principle

We now introduce the adjoint equation. Recall the notation $\theta_t^u = (X_t^u, u_t)$, where X^u denotes the unique solution to the controlled state equation associated with the control $u \in \mathcal{U}'_{ad}$.

The *adjoint equation* is the following BSDE:

$$\begin{cases} \mathrm{d}p_t = \left[-p_t^T D_x b(t,\theta_t^u) - \mathrm{Tr}(q_t^T D_x \sigma(t,\theta_t^u)) - D_x f(t,\theta_t^u)\right] \mathrm{d}t + q_t \mathrm{d}W_t \\ p_T = D_x h(X_T^u). \end{cases} \tag{3.8}$$

where the pair (p_t, q_t) are an unknown adapted pair of processes taking values in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$ respectively.

To simplify notation, we introduce the *Hamiltonian*

$$\begin{aligned} H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times d} &\to \mathbb{R}, \\ H(t,x,u,p,q) &:= \langle b(t,x,u), p\rangle + \langle \sigma(t,x,u), q\rangle + f(t,x,u) \\ &= p^T b(t,x,u) + \mathrm{Tr}(q^T\sigma(t,x,u)) + f(t,x,u). \end{aligned}$$

In terms of H, the adjoint equation can be compactly written as

$$\begin{cases} \mathrm{d}p_t = -D_x H(t, X_t^u, u_t, p_t, q_t)\mathrm{d}t + q_t \mathrm{d}W_t, \\ p_T = D_x h(X_T^u). \end{cases} \tag{3.9}$$

The tuple (p_t, q_t) is to be intended as a solution to (3.9) in the sense of Definition 2.1.

We make one last assumption which will be needed for the Pontryagin maximum principle.

Assumptions ($\mathcal{P}$-3). *The maps* $u \mapsto H(\cdot,\cdot,u,\cdot,\cdot)$ *convex.*

Note that if the maps $u \mapsto f(\cdot,\cdot,u)$ is convex and the coefficients b and σ are additive in the control then the Assumption ($\mathcal{P}$-3) are satisfied.

Theorem 3.4 (Well-posedness of the adjoint equation)**.** Let assumptions ($\mathcal{P}$-**1**) and ($\mathcal{P}$-**2**) hold. Then, there exists a unique solution $(p, q) \in \mathbb{S}^2(\mathbb{R}^n) \times \mathbb{H}^2(\mathbb{R}^{n\times d})$ to the adjoint equation (3.9) in the sense of Definition 2.1.

Proof. The result follows from standard existence and uniqueness theorems for BSDEs once the assumptions of Assumptions ($\mathcal{H}$-**2**) are verified.

1. **Terminal condition.** $D_x h(X_T^u) \in L^2(\Omega; \mathbb{R}^n)$ which is true since $D_x h$ is $\mathcal{F}_T$-adapted by assumption and

$$D_x h(X_T^u) \leq C|X_T^u|^{q-1} \leq +\infty,$$

since we know that $X \in L^p$ for all $p > 1$ and $q \geq 1$.

2. **Integrability of the generator at zero.** The process $\{D_x H(t, \theta_t^u, 0, 0), t \in [0, T]\}$ satisfies

$$D_x H(t, \theta_t^u, 0, 0) = D_x f(t, \theta_t^u),$$

which is assumed to be have polynomial growth by assumption. Hence

$$\mathbb{E}\left[\int_0^T |D_x f(t, \theta_t^u)| \mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T C(1 + |X_t^u|^q + |u_t|^q) \mathrm{d}t\right] < +\infty,$$

since $X \in L^p(\Omega \times [0, T]; \mathbb{R}^n)$ and $u \in L^p(\Omega \times [0, T]; \mathbb{R}^k)$ and $p > 2q$.

3. **Lipschitz property.** For any $(p, q), (p', q') \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, by linearity of the inner product and of the trace, by the triangle inequality 1.3, there exists $L > 0$ such that

$$\begin{aligned} |D_x H(t, \theta_t^u, p, q) - &D_x H(t, \theta_t^u, p', q')| \\ &= \left|(p - p')^\top D_x b(t, \theta_t^u) + \mathrm{Tr}\big((q - q')^\top D_x \sigma(t, \theta_t^u)\big)\right| \\ &\leq L\left(|p - p'| + |q - q'|\right) \end{aligned}$$

which shows that the generator is Lipschitz in (p, q).

Since all conditions of Assumptions ($\mathcal{H}$-**2**) are satisfied, by Theorem 2.1, there exists a unique adapted pair (p, q) solving (3.9). □

3.4.1 Representation of the Gradient of the Cost Functional

We now represent the gradient of the cost functional in terms of the solution to the adjoint equation.

Proposition 3.3. Let assumptions ($\mathcal{P}$-**1**) and ($\mathcal{P}$-**2**) hold. Let $u \in \mathcal{U}'_{ad}$, $h \in \mathcal{U}$, let X^u the solution to the state equation (3.1), let Y^h the solution of the linearized equation (3.7), and let (p,q) be the solution to the adjoint equation (3.8). Then, the following identity holds:

$$\mathbb{E}\left[\langle p_T, Y_T^h\rangle\right] = \mathbb{E}\left[\int_0^T \Big(\langle p_t, D_u b(t,\theta_t)[h]\rangle + \langle q_t, D_u\sigma(t,\theta_t)[h]\rangle - D_x f(t,\theta_t)[Y_t^h]\Big)\mathrm{d}t\right].$$

Proof. By Itô's formula 1.11

$$\mathrm{d}\langle p_t, Y_t^h\rangle = \langle \mathrm{d}p_t, Y_t^h\rangle + \langle p_t, \mathrm{d}Y_t^h\rangle + \mathrm{d}[p_t, Y_t^h].$$

Substituting the dynamics of p_t and Y_t^h, we obtain:

1. From the adjoint BSDE (3.8),

$$\langle \mathrm{d}p_t, Y_t^h\rangle = -\Big(\langle D_x b(t,\theta_t)[Y_t^h], p_t\rangle + \langle D_x\sigma(t,\theta_t)[Y_t^h], q_t\rangle \\ + D_x f(t,\theta_t)[Y_t^h]\Big)\mathrm{d}t + \langle q_t \mathrm{d}W_t, Y_t^h\rangle,$$

2. From the linearized SDE (3.7),

$$\langle p_t, \mathrm{d}Y_t^h\rangle = \Big(\langle p_t, D_x b(t,\theta_t)[Y_t^h] + D_u b(t,\theta_t)[h]\rangle\Big)\mathrm{d}t \\ + \langle p_t, D_x\sigma(t,\theta_t)[Y_t^h] + D_u\sigma(t,\theta_t)[h]\rangle \mathrm{d}W_t,$$

3. Finally,

$$\mathrm{d}[p_t, Y_t^h] = \langle q_t, D_x\sigma(t,\theta_t^u)[Y_t^h] + D_u\sigma(t,\theta_t^u)[h]\rangle \mathrm{d}t.$$

Collecting all terms, we find:

$$\mathrm{d}\langle p_t, Y_t^h\rangle = \Big(\langle p_t, D_u b(t,\theta_t)[h]\rangle + \langle q_t, D_u\sigma(t,\theta_t)[h]\rangle - D_x f(t,\theta_t)[Y_t^h]\Big)\mathrm{d}t \\ + \underbrace{\left(\langle p_t, D_x\sigma(t,\theta_t)[Y_t^h] + D_u\sigma(t,\theta_t)[h]\rangle\right)\mathrm{d}W_t}_{M_1} + \underbrace{\langle q_t \mathrm{d}W_t, Y_t^h\rangle}_{M_2}.$$

M_1 and M_2 are local martingales. Indeed, by boundedness of the derivatives of σ and the square-integrability of p^n, q^n, and Y^v, there exists $C > 0$ such that

$$|M_1|^2 \leq |p_t|^2 \cdot |D_x\sigma(t,\theta_t)[Y_t^h] + D_u\sigma(t,\theta_t)[h]|^2, \\ |M_2|^2 \leq |q_t|^2 \cdot |Y_t^h|^2,$$

Now, note that

$$|D_x\sigma(t,\theta_t)[Y_t^h] + D_u\sigma(t,\theta_t)[h]| \leq |D_x\sigma(t,\theta_t)[Y_t^h]| + |D_u\sigma(t,\theta_t)[h]| \\ \leq C(|X_t^u| + |u_t|)(|Y_t^h| + |h|) \leq 2C'(|X_t^u| + |u_t|)|h|$$

and since $X \in \mathcal{X}$ and $u \in \mathcal{U}_{ad}$, both M_1, M_2 are true martingales with zero expectation.

Integrating between 0 and T, taking expectations, and using $\mathbb{E}[\langle p_0, Y_0^h\rangle] = 0$, we obtain the thesis

$$\mathbb{E}\left[\langle p_T, Y_T^h\rangle\right] = \mathbb{E}\left[\int_0^T \left(\langle p_t, D_u b(t,\theta_t)[h]\rangle + \langle q_t, D_u\sigma(t,\theta_t)[h]\rangle - D_x f(t,\theta_t)[Y_t^h]\right)\mathrm{d}t\right].$$

since $\mathbb{E}\left[\langle p_0, Y_0^h\rangle\right] = 0$. □

Theorem 3.5 (Representation of the Gradient)**.** Let Assumptions ($\mathcal{P}$-**1**), ($\mathcal{P}$-**2**), ($\mathcal{P}$-**3**) hold. Then, for $u \in \mathcal{U}'_{ad}$ and X^u the solution to the state equation (3.1), and let (p,q) be the solution to the adjoint equation (3.8), for every $h \in \mathcal{U}$, it holds

$$D\widetilde{J}(u)[h] = \mathbb{E}\left[\int_0^T D_u H(t, X_t^u, u_t, p_t, q_t)[h]\mathrm{d}t\right].$$

Proof. From the terminal condition of the adjoint equation (3.8), we have

$$\mathbb{E}\left[D_x h(X_T^u)[Y_T^h]\right] = \mathbb{E}\left[\langle D_x h(X_T^u), Y_T^h\rangle\right] = \mathbb{E}\left[\langle p_T, Y_T^h\rangle\right].$$

Using Proposition 3.3, we can write

$$D\widetilde{J}(u)[h] = \mathbb{E}\left[\int_0^T \left(D_u f(t,\theta_t^u)[h_t] + \langle p_t, D_u b(t,\theta_t^u)[h_t]\rangle + \langle q_t, D_u\sigma(t,\theta_t^u)[h_t]\rangle\right)\mathrm{d}t\right].$$

Observing that

$$D_u H(t, X_t^u, u_t, p_t, q_t)[h] = \langle p_t, D_u b(t,\theta_t)[h]\rangle + \langle q_t, D_u\sigma(t,\theta_t)[h]\rangle + D_u f(t,\theta_t)[h],$$

the claim follows immediately. □

3.4.2 The Pontryagin Maximum Principle

We can finally state the Pontryagin Maximum Principle.

Theorem 3.6 (Pontryagin Maximum Principle)**.** Let Assumptions ($\mathcal{P}$-**1**), ($\mathcal{P}$-**2**) hold. Let $\overline{u} \in \mathcal{U}'_{ad}$ be an optimal control for the problem. If $\overline{X}$ is the solution to the state equation (3.1), and let (p,q) be the solution to the corresponding adjoint equation (3.8). Then, for almost every $t \in [0,T]$ for all $u \in \mathcal{U}'_{ad}$

$$\mathbb{E}\left[\int_0^T D_u H(t, \overline{X}_t, \overline{u}_t, p_t, q_t)[u_t - \overline{u}_t]\,\mathrm{d}t\right] \geq 0.$$

In addition if Assumption ($\mathcal{P}$-3) hold then we have

$$H(t, \overline{X}_t, \overline{u}, p_t, q_t) \leq H(t, \overline{X}_t, u, p_t, q_t), \quad \mathbb{P} - \text{a.s. for a.e. } t \in [0, T].$$

Proof. Since $\mathcal{U}_{ad} \subset L^2(\Omega \times [0, T]; \mathbb{R}^k)$ is convex, for any control $u \in \mathcal{U}_{ad}$ and any $\epsilon \in (0, 1)$, the convex combination

$$u^\epsilon := (1 - \epsilon)\overline{u} + \epsilon u$$

also belongs to $\mathcal{U}_{ad}$.

By the Gâteaux differentiability of the reduced cost functional,

$$\widetilde{J}(u^\epsilon) = \widetilde{J}(\overline{u}) + \epsilon\, D\widetilde{J}(\overline{u})[u - \overline{u}] + o(\epsilon).$$

and since $\overline{u}$ is optimal, $\widetilde{J}(u^\epsilon) - \widetilde{J}(\overline{u}) \geq 0$ for all $\epsilon > 0$. Dividing by ϵ and letting $\epsilon \to 0^+$, we obtain

$$D\widetilde{J}(\overline{u})[u - \overline{u}] \geq 0,$$

Using the gradient representation proved in Theorem 3.5, we can write

$$D\widetilde{J}(\overline{u})[u - \overline{u}] = \mathbb{E}\left[\int_0^T D_u H(t, \overline{X}_t, \overline{u}_t, p_t, q_t)[u_t - \overline{u}_t]\, \mathrm{d}t\right] \geq 0.$$

Now if Assumption ($\mathcal{P}$-3) hold, then $H(t, x, \cdot, p, q)$ is convex in u, i.e.

$$H(\overline{u}) + \langle D_u H(\overline{u}), u - \overline{u}\rangle \leq H(u), \quad \forall u \in \mathcal{U}_{ad},$$

so that by integrating in time and taking the expectation we deduce

$$\mathbb{E}\left[\int_0^T H(t, \overline{X}_t, \overline{u}_t, p_t, q_t)\mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T H(t, \overline{X}_t, u_t, p_t, q_t)\mathrm{d}t\right].$$

By a classical localization argument this implies that

$$H(t, \overline{X}_t, \overline{u}_t, p_t, q_t) \leq H(t, \overline{X}_t, u_t, p_t, q_t) \quad \mathbb{P} - \text{a.s. for a.e. } t \in [0, T].$$

□

4

Pontryagin's Maximum Principle: random final time

In classical stochastic optimal control problems, the terminal time T is fixed and known in advance. However, in many practical situations, such as option exercise decisions in finance or resource depletion timing, it is natural to allow the terminal time to be part of the control itself. This approach is particularly useful in finance, where decisions are often made based on the state of the market or the performance of an investment portfolio.

An important class of problems where this setting is used is the *optimal stopping* problems, which knows a renewed interest due to its numerous and various applications.

4.1 Problem Formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a complete filtered probability space, where $(\mathcal{F}_t)_t$ is the augmented natural filtration generated by a d-dimensional Brownian motion W. We consider the controlled stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \sigma(t, X_t, u_t)\mathrm{d}W_t, & t \in (0, T] \\ X_0 = x_0 \in \mathbb{R}^n \end{cases}$$

where $u \in \mathcal{U}_{ad}$, with $U \subseteq \mathbb{R}^k$ a convex set and let $p > 2$:

$$\mathcal{U}_{ad} := \{u : \Omega \times [0, T] \to U : (\mathcal{F}_t)_t\text{-progressively measurable and } u \in L^p(\Omega \times [0, T]; \mathbb{R}^k)\}.$$

We now allow the evaluation of the state equation up to some stopping time τ, where:

$$\tau \in \mathcal{S}_T := \{\tau : \Omega \to [0,T] \text{ is an } \mathcal{F}_t\text{-stopping time}\}.$$

We assume $0 < \tau \leq T$ $\mathbb{P}$-a.s. Note that, in this framework, the control is the pair (u, τ).

The cost functional is defined as

$$\begin{aligned} J &: \mathcal{X} \times \mathcal{U}_{ad} \times \mathcal{S}_T \to \mathbb{R} \\ J(x,u,\tau) &:= \mathbb{E}\left[\int_0^\tau f(t,\tau,X_t^u,u_t)\mathrm{d}t\right] \end{aligned} \tag{4.1}$$

where $f : [0,T]^2 \times \mathbb{R}^n \times U \to \mathbb{R}$ is the running cost function. We removed the final cost, for now, since it makes for an easier initial formulation, it will be introduced again in the next section. Observe that X^u up to time τ is equal to the dynamics of the solution of the same SDE up to time T but considered until time τ. Hence, the control-to-state map

$$\begin{aligned} S &: \mathcal{U}_{ad} \times \mathcal{S}_T \to \mathcal{X}, \\ S(u,\tau) &= X^u. \end{aligned}$$

This is independent of the stopping time τ. Therefore, its differentiability with respect to u is unchanged to the analysis done in Section 3.3.

The stochastic optimal control problem with random terminal time is then formulated as

$$\inf_{(u,\tau)\in\mathcal{U}_{ad}\times\mathcal{S}_T} \widetilde{J}(u,\tau), \qquad \text{where} \quad \widetilde{J}(u,\tau) := J(X^u,u,\tau).$$

We seek an optimal pair $(\overline{u}, \overline{\tau})$ such that

$$\widetilde{J}(\overline{u},\overline{\tau}) = \inf_{(u,\tau)} \widetilde{J}(u,\tau).$$

Since f is not in the same setting as it was in Section 3.1.1 we need to state a new version of it:

Assumptions ($\mathcal{P}'$-2). *The function f satisfy:*

1. *For every $u \in \mathbb{R}^k$, the map $(t, \tau, x) \mapsto f(t, \tau, x, u)$ is measurable with respect to $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{S}_T$ and is C^1.*
2. *There exists $C > 0$ and $q \in [1, \frac{p}{2})$ such that for every $(x, u) \in \mathbb{R}^n \times U$, for every $\tau \in \mathcal{S}_T$ and for all $t \in [0,T]$*

$$\begin{aligned} |D_x f(t,\tau,x,u)| &\leq C(1 + |x|^q + |u|^q + |\tau|^q) \\ |D_u f(t,\tau,x,u)| &\leq C(1 + |x|^q + |u|^q + |\tau|^q) \\ |D_\tau f(t,\tau,x,u)| &\leq C(1 + |x|^q + |u|^q + |\tau|^q) \end{aligned}$$

4.1.1 Linearization and Variational Equations

In the same fashion as the introduction of $\mathcal{U}$ we need to introduce the Banach space $\mathcal{S} := L^2(\Omega)$ as the space where the directions of the derivatives are taken with $\mathcal{S}_T \subseteq \mathcal{S}$. We denote $\theta_t = (X_t^u, u_t)$.

Theorem 4.1. If Assumptions ($\mathcal{P}$-1) and ($\mathcal{P}'$-2) hold. Let $u \in \mathcal{U}'_{ad}$, $v \in \mathcal{U}$, let $\tau \in \mathcal{S}_T$ and $\rho \in \mathcal{S}$, let X^u be the solution to the state equation (4.1), let Y^v be the solution of the linearized equation

$$\begin{cases} \mathrm{d}Y_t^v = \Big(D_x b(t,\theta_t)[Y_t^v] + D_u b(t,\theta_t)[v]\Big)\mathrm{d}t + \Big(D_x \sigma(t,\theta_t)[Y_t^v] + D_u \sigma(t,\theta_t)[v]\Big)\mathrm{d}W_t, \\ Y_0^v = 0. \end{cases} \tag{4.2}$$

Then, the adjoint equation

$$\mathrm{d}p_t = -\Big(D_x f(t,\tau,\theta_t) + D_x b(t,\theta_t)[p_t] + D_x \sigma(t,\theta_t)[q_t]\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \qquad p_\tau = 0. \tag{4.3}$$

admits a unique solution (p, q) in the sense of Theorem 2.2 and it holds

$$D\widetilde{J}(u,\tau)[v,\rho] = \mathbb{E}\left[\int_0^\tau \Big(D_u \mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t)[v] + D_\tau \mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t)[\rho]\Big)\mathrm{d}t\right],$$

where

$$\begin{gathered} \mathcal{H} : [0,T]^2 \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}, \\ \mathcal{H}(t,\tau,x,u,p,q) := \mathbb{1}_{\{t\leq\tau\}} f(t,\tau,x,u) + \langle b(t,x,u), p\rangle + \mathrm{Tr}(\sigma(t,x,u)q^\top). \end{gathered}$$

Note that the $D_\tau \mathcal{H}$ is to be intended as the distributional derivative. Moreover, when $\tau = 0$ or $\tau = T$ we implicitly assume that the increment ρ is pointing in the interior of $[0, T]$.

Proof. Let $(u, \tau) \in \mathcal{U}'_{ad} \times \mathcal{S}_T$ denote the an admissible pair. We introduce perturbations in both the control and the stopping time:

$$u^\epsilon := u + \epsilon v, \qquad \tau^\epsilon := \tau + \epsilon \rho,$$

where $v \in \mathcal{U}$ and $h \in \mathcal{S}$, and $\epsilon > 0$ sufficiently small to ensure that $\tau^\epsilon \in \mathcal{S}_T$ and $u + \epsilon v \in \mathcal{U}_{ad}$.

The linearized state equation corresponding to the perturbation in u is given by (4.2). To handle the dependence of the cost on τ, it is convenient to introduce the auxiliary process

$$\mathrm{d}Z_t := \mathbb{1}_{\{t \leq \tau\}} f(t, \tau, \theta_t) \mathrm{d}t,$$

so that $\widetilde{J}(u, \tau) = \mathbb{E}[Z_T]$.

The first variation of Z_T with respect to (u, τ) is obtained as

$$\frac{Z_T(u^\epsilon, \tau^\epsilon) - Z_T(u, \tau)}{\epsilon} = I_1^\epsilon + I_2^\epsilon,$$

where

$$I_1^\epsilon = \int_0^\tau \frac{f(t, \tau + \epsilon h, X_t^{u^\epsilon}, u_t^\epsilon) - f(t, \tau, X_t^u, u_t)}{\epsilon} \, \mathrm{d}t,$$
$$I_2^\epsilon = \frac{1}{\epsilon} \int_\tau^{\tau + \epsilon h} f(t, \tau + \epsilon h, X_t^{u^\epsilon}, u_t^\epsilon) \, \mathrm{d}t,$$

denoting X_t^u the solution of the state equation (4.1).

Under Assumptions ($\mathcal{P}'$-2), by a Taylor expansion we have

$$f(t, \tau + \epsilon \rho, X_t^{u^\epsilon}, u_t^\epsilon) = f(t, \tau, X_t^u, u_t) + \epsilon \Bigg(D_\tau f(t, \tau, X_t^u, u_t) \rho + D_x f(t, \tau, X_t^u, u_t) \left[\frac{X_t^{u^\epsilon} - X_t^u}{\epsilon} \right] + D_u f(t, \tau, X_t^u, u_t) \left[\frac{u_t^\epsilon - u_t}{\epsilon} \right] \Bigg) + o(\epsilon).$$

Now $\frac{X_t^{u^\epsilon} - X_t^u}{\epsilon} \to Y_t^h$ by Section 3.3, and $\frac{u_t^\epsilon - u_t}{\epsilon} \to v$. Thus, since $X^u \in L^p$ and by bounded growth as stated in Assumptions ($\mathcal{P}'$-2), by dominated convergence theorem we have

$$\lim_{\epsilon\to 0} I_1^\epsilon = \int_0^\tau \big(D_\tau f(t,\tau,X_t^u,u_t)\rho + D_x f(t,\tau,X_t^u,u_t)[Y_t^h] + D_u f(t,\tau,X_t^u,u_t)[v]\big)\,\mathrm{d}t.$$

Consider I_2^ϵ, the first-order expansion of the integrand in (t,τ,X_τ^u,u_τ) is

$$\begin{aligned} f(t,\tau+\epsilon\rho,X_t^{u^\epsilon},u_t^\epsilon) = f(\tau,\tau,X_\tau^u,u_\tau) + A_1(t-\tau) + A_2(\epsilon\rho) + A_3(X_t^{u^\epsilon} - X_\tau^u) + \\ A_4(u_t^\epsilon - u_\tau) + o(|t-\tau| + |\epsilon\rho| + |X_t^{u^\epsilon} - X_\tau^u| + |u_t^\epsilon - u_\tau|). \end{aligned}$$

Integrating over $[\tau,\tau+\epsilon\rho]$ and dividing by ϵ the terms linear in $t-\tau$ and in $\epsilon\rho$ will generate contributions $O(\epsilon)$, while state and control variations generate $O(\sup_{[\tau,\tau+\epsilon h]}|X_t^{u^\epsilon} - X_\tau^u|)$ and same for u. By an application of Lemma 3.1 we have that $\sup_{[\tau,\tau+\epsilon\rho]}|X_t^{u^\epsilon} - X_\tau^u| = O(\epsilon)$. Hence, we have

$$I_2^\epsilon = \frac{1}{\epsilon}\epsilon\rho f(\tau,\tau,X_\tau^u,u_\tau) + O(\epsilon) = \rho f(\tau,\tau,X_\tau^u,u_\tau) + O(\epsilon),$$

Putting things together we get:

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{Z_T(u^\epsilon,\tau^\epsilon) - Z_T(u,\tau)}{\epsilon} &= \int_0^\tau \big(\rho D_\tau f(t,\tau,\theta_t) + D_x f(t,\tau,\theta_t)[Y_t^v] + D_u f(t,\tau,\theta_t)[v]\big)\mathrm{d}t \\ &\quad + \rho\, f(\tau,\tau,X_\tau^u,u_\tau), \end{aligned} \tag{4.4}$$

The term involving $f(\tau,\tau,\cdot)$ in (4.4) introduces a discontinuity at $t=\tau$. To handle this rigorously, we approximate the indicator function $\mathbb{1}_{\{t\le\tau\}}$ by a family of smooth functions

$$\chi_n : [0,T]^2 \to [0,1]$$

such that

$$\chi_n(t,\tau) \to \mathbb{1}_{\{t\le\tau\}}, \qquad D_\tau\chi_n(t,\tau) \to \delta(t-\tau),$$

where δ denotes the Dirac delta distribution. The map $t\mapsto\chi_n(t,\cdot)$ is required to be $[0,T]-\mathcal{B}([0,T])$ measurable. The function χ_n is zero for values of $t<t'<\tau$ and is one for values of $t>t''>\tau$.

Define the *smoothed auxiliary process* as

$$\mathrm{d}Z_t^n = \chi_n(t,\tau)\,f(t,\tau,\theta_t)\,\mathrm{d}t, \qquad Z_0^n = 0,$$

and the corresponding approximated cost functional

$$\widetilde{J}^n(u,\tau) = \mathbb{E}[Z_T^n].$$

The first variation is given by

$$D\widetilde{J}^n(u,\tau)[v,\rho] = \mathbb{E}\Bigg[\int_0^T \chi_n(t,\tau)\big(D_x f(t,\tau,\theta_t)[Y_t^v] + D_u f(t,\tau,\theta_t)[v_t] + \rho D_\tau f(t,\tau,\theta_t)\big) \\ + D_\tau \chi_n(t,\tau)\,\rho\, f(t,\tau,\theta_t)\,\mathrm{d}t\Bigg].$$

Let (p^n, q^n) be the adjoint pair solving the smoothed BSDE.

$$\mathrm{d}p_t^n = -\Big(D_x b(t,\theta_t)[p_t^n] + D_x\sigma(t,\theta_t)[q_t^n] + \chi_n(t,\tau)D_x f(t,\tau,\theta_t)\Big)\,\mathrm{d}t + q_t^n\,\mathrm{d}W_t, \quad p_\tau^n = 0. \tag{4.5}$$

We extend this solution to the horizon $[0,T]$ by setting $p_t^n \equiv 0$ and $q_t^n \equiv 0$ for every $t > \tau$. Indeed, we can extend p_t^n at 0 by continuity of the solution of (4.5), if we want 0 to solve the same equation we require that both the right and left-hand side are identically zero.

- $\chi_n(t,\tau)D_x f(t,\tau,\theta_t) \equiv 0$ for all $t \geq \tau$ by the definition of χ_n.
- $D_x b(t,\theta_t)[p_t^n] \equiv 0$ since we have set $p_t^n = 0$ for all $t \geq \tau$.

From this we conclude that $q_t^n \equiv 0$ for all $t \geq \tau$. With the above extension at 0 on $[\tau, T]$, (p,q) still resolve the adjoint equation (4.5) up to T.

By Itô's formula 1.11

$$\mathrm{d}\langle p_t^n, Y_t^v\rangle = \langle \mathrm{d}p_t^n, Y_t^v\rangle + \langle p_t^n, \mathrm{d}Y_t^v\rangle + \mathrm{d}[p_t^n, Y_t^v].$$

Substituting the dynamics of p_t^n and Y_t^v, we obtain:

1. From the adjoint BSDE (4.5),

$$\langle \mathrm{d}p_t^n, Y_t^v\rangle = -\Big(\langle D_x b(t,\theta_t)[Y_t^v], p_t^n\rangle + \langle D_x\sigma(t,\theta_t)[Y_t^v], q_t^n\rangle \\ + \chi_n(t,\tau)D_x f(t,\tau,\theta_t)[Y_t^v]\Big)\mathrm{d}t \\ + \langle q_t^n \mathrm{d}W_t, Y_t^v\rangle,$$

2. From the linearized SDE (3.7),

$$\mathrm{d}[p_t^n, Y_t^v] = \Big(\langle p_t^n, D_x b(t,\theta_t)[Y_t^v] + D_u b(t,\theta_t)[v]\rangle\Big)\mathrm{d}t \\ + \langle p_t^n, D_x\sigma(t,\theta_t)[Y_t^v] + D_u\sigma(t,\theta_t)[v]\rangle \mathrm{d}W_t,$$

3. Finally,

$$\langle \mathrm{d}p_t^n, dY_t^v\rangle = \langle q_t, D_x\sigma(t,\theta_t)[Y_t^v] + D_u\sigma(t,\theta_t)[v]\rangle\mathrm{d}t.$$

Collecting all terms, we find:

$$d\langle p_t^n, Y_t^v\rangle = \Big(\langle p_t^n, D_u b(t,\theta_t)[v]\rangle + \langle q_t^n, D_u\sigma(t,\theta_t)[v]\rangle - \chi_n(t,\tau)D_x f(t,\tau,\theta_t)[Y_t^v]\Big)\mathrm{d}t \\ + \underbrace{\langle p_t^n, D_x\sigma(t,\theta_t)[Y_t^v] + D_u\sigma(t,\theta_t)[v]\rangle\mathrm{d}W_t}_{M_1} + \underbrace{\langle q_t^n \mathrm{d}W_t, Y_t^v\rangle}_{M_2}.$$

M_1 and M_2 are local martingales. Indeed, by boundedness of the derivatives of σ and the square-integrability of p^n, q^n, and Y^v, there exists $C > 0$ such that

$$|M_1|^2 \leq |p_t^n|^2 \cdot |D_x\sigma(t,\theta_t)[Y_t^v] + D_u\sigma(t,\theta_t)[v]|^2, \\ |M_2|^2 \leq |q_t^n|^2 \cdot |Y_t^v|^2,$$

Now, note that

$$|D_x\sigma(t,\theta_t)[Y_t^v] + D_u\sigma(t,\theta_t)[v]| \leq |D_x\sigma(t,\theta_t)[Y_t^v]| + |D_u\sigma(t,\theta_t)[v]| \\ \leq C(|X_t^u| + |u_t|)(|Y_t^v| + |v|) \leq 2C(|X_t^u| + |u_t|)|v|$$

and since $X \in \mathcal{X}$ and $u \in \mathcal{U}_{ad}$, both M_1, M_2 are true martingales with zero expectation.

Integrating between 0 and T, taking expectations, and using $\mathbb{E}[\langle p_0^n, Y_0^v\rangle] = 0$ we obtain the thesis

$$\mathbb{E}\left[(p_\tau^n)^T Y_\tau^v\right] = \mathbb{E}\Bigg[\int_0^\tau \Big(\langle p_t^n, D_u b(t,\theta_t)[v]\rangle + \langle q_t^n, D_u\sigma(t,\theta_t)[v]\rangle \\ - \chi_n(t,\tau)D_x f(t,\theta_t)[Y_t^v]\Big)\mathrm{d}t\Bigg].$$

Since $\mathbb{E}\left[\langle p^n_\tau, Y_\tau\rangle\right] = 0$, the result becomes

$$\mathbb{E}\left[\int_0^\tau \chi_n(t,\tau) D_x f(t,\tau\theta_t)[Y^v_t]\mathrm{d}t\right] \\ = \mathbb{E}\left[\int_0^\tau \Big(\langle p^n_t, D_u b(t,\theta_t)[v]\rangle + \langle q^n_t, D_u \sigma(t,\theta_t)[v]\rangle\Big)\mathrm{d}t\right].$$

Note that since we have extended the solution of the approximated adjoint equation (4.5) the result is valid also on $[0,T]$

$$\mathbb{E}\left[\int_0^T \chi_n(t,\tau) D_x f(t,\tau,\theta_t)[Y^v_t]\mathrm{d}t\right] \\ = \mathbb{E}\left[\int_0^T \Big(\langle p^n_t, D_u b(t,\theta_t)[v]\rangle + \langle q^n_t, D_u \sigma(t,\theta_t)[v]\rangle\Big)\mathrm{d}t\right].$$

We define the *approximated extended Hamiltonian*

$$\mathcal{H}^n : [0,T]^2 \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}, \\ \mathcal{H}^n(t,\tau,x,u,p,q) := \chi_n(t,\tau)\, f(t,\tau,x,u) + p^T b(t,x,u) + \mathrm{Tr}[\sigma(t,x,u)q^\top].$$

From the terminal condition of the adjoint equation (4.5), we have

$$\mathbb{E}\left[\langle p^n_T, Y^v_T\rangle\right] = 0$$

Using Proposition 4.1, we can write

$$D\widetilde{J}^n(u,\tau)[v,\rho] \\ = \mathbb{E}\left[\int_0^T \chi_n(t,\tau)\big(D_x f(t,\tau,\theta_t)[Y^v_t] + D_u f(t,\tau,\theta_t)[v] + \rho D_\tau f(t,\tau,\theta_t)\big)\,\mathrm{d}t\right].$$

Observing that

$$D_u\mathcal{H}^n(t,\tau,X^u_t,u_t,p_t,q_t)[v] = \langle p^n_t, D_u b(t,\tau,\theta_t)[v]\rangle + \langle q^n_t, D_u\sigma(t,\theta_t)[v]\rangle \\ + \chi_n(t,\tau) D_u f(t,\tau,\theta_t)[v], \\ D_\tau\mathcal{H}^n(t,\tau,X^u_t,u_t,p_t,q_t)[\rho] = \chi_n(t,\tau) D_\tau f(t,\tau,\theta_t)[\rho] + D_\tau\chi_n(t,\tau) f(t,\tau,\theta_t)[\rho].$$

Hence,

$$D\widetilde{J}(u,\tau)[v,\rho] = \mathbb{E}\left[\int_0^T \left(D_u\mathcal{H}^n(t,\tau,X_t^u,u_t,p_t^n,q_t^n)[v] + D_\tau\mathcal{H}^n(t,\tau,X_t^u,u_t,p_t^n,q_t^n)[\rho]\right)\mathrm{d}t\right].$$

Passing to the limit as $n \to \infty$ is, in general, straightforward. The only delicate point concerns the weak derivative of the indicator function, which is the Dirac delta. Recall that

$$\frac{d}{dx}H(x) = \delta(x),$$

where H is the Heaviside step function.

Consequently, the limit *adjoint equation* takes the form, formally

$$\mathrm{d}p_t = -\Big(D_x f(t,\tau,\theta_t) + D_x b(t,\theta_t)[p_t] + D_x\sigma(t,\theta_t)[q_t]\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \qquad p_\tau = 0. \tag{4.6}$$

The existence and uniqueness of the solution of this BSDE follow from Theorem 3.4. Indeed, one can follow the same proof and noting that the generator in this case is dominated by the one considered in the proof.

Moreover, as before, we extend the solution to the whole interval $[0,T]$ by setting $p_t = q_t \equiv 0$ for $t > \tau$. Then these extended adjoint equation follows the following BSDE:

$$\mathrm{d}p_t = -\Big(\mathbb{1}_{\{t\le\tau\}}D_x f(t,\tau,\theta_t) + D_x b(t,\theta_t)[p_t] + D_x\sigma(t,\theta_t)[q_t]\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \qquad p_T = 0. \tag{4.7}$$

Since the generators

$$-\Big(D_x b(t,\tau,\theta_t)[p_t^n] + D_x\sigma(t,\tau,\theta_t)[q_t^n] + \chi_n(t,\tau)D_x f(t,\tau,\theta_t)\Big)$$
$$\to -\Big(\mathbb{1}_{\{t\le\tau\}}D_x f(t,\tau,\theta_t) + D_x b(t,\theta_t)[p_t] + D_x\sigma(t,\theta_t)[q_t]\Big)$$

in $L^2(\Omega\times[0,T])$, then the sequence of solutions $(p^n,q^n)\to(p,q)$ in $\mathbb{S}_T^2\times\mathbb{H}_T^2$; see Jia and Yu (2008) and Zhang (2010) for a proof of this result.

The *extended Hamiltonian* in the limit takes the form

$$\mathcal{H} : [0,T]^2\times\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^{n\times n}\to\mathbb{R},$$
$$\mathcal{H}(t,\tau,X^u,u,p,q) := \mathbb{1}_{\{t\le\tau\}}f(t,\tau,X_t^u,u) + \langle b(t,X_t^u,u),p\rangle + \mathrm{Tr}(\sigma(t,X_t^u,u)q^\top).$$

In particular, we have pointwise

$$D_u\mathcal{H}^n(t,\tau,X_t^u,u_t,p_t^n,q_t^n) \to D_u\mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t),$$
$$D_\tau\mathcal{H}^n(t,\tau,X_t^u,u_t,p_t^n,q_t^n) \to D_\tau\mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t),$$

where

$$D_u\mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t) := \mathbb{1}_{\{t\le\tau\}}D_uf(t,\tau,X_t^u,u_t) + D_ub(t,\theta_t)[p_t] + D_u\sigma(t,\theta_t)[q_t],$$
$$D_\tau\mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t) := \mathbb{1}_{\{t\le\tau\}}D_\tau f(t,\tau,X_t^u,u_t) + \delta(t-\tau)f(t,\tau,X_t^u,u_t).$$

Then, by the dominated convergence theorem, we obtain the thesis. □

4.2 Adding the Final Cost

In this section, we extend the stochastic optimal control problem with random terminal time τ by including a final cost depending on the state at the stopping time. The new cost functional is defined as

$$\widetilde{J}(u,\tau) = \mathbb{E}\left[\int_0^\tau f(t,\tau,X_t^u,u_t)\mathrm{d}t + h(X_\tau^u)\right], \tag{4.8}$$

where $h:\mathbb{R}^n\to\mathbb{R}$ is a smooth function representing the terminal payoff and X^u is the solution of the state equation (4.1). We will derive the Gâteaux derivative of this functional with respect to both the control and the stopping time τ.

Further assumptions on h are needed:

Assumptions ($\mathcal{P}'$-3). *The function h satisfies:*

1. *h is measurable with respect to $\mathcal{B}(\mathbb{R}^n)$.*
2. *h is C^2.*
3. *There exists a constant $C>0$ and $q\in[1,\frac{p}{2})$ such that for every $x\in\mathbb{R}^n$:*

$$|h(x)| \le C(1+|x|^q).$$

4.2.1 Gâteaux differentiability of the Cost Functional

Theorem 4.2. Let Assumptions ($\mathcal{P}$-1), ($\mathcal{P}'$-2) and ($\mathcal{P}'$-3) hold. Let $u \in \mathcal{U}'_{ad}$, $v \in \mathcal{U}$, let $\tau \in \mathcal{S}_T$ and $\rho \in \mathcal{S}$, let X^u be the solution to the state equation (4.1), let Y^v be the solution of the linearized equation (4.2). Then the Gâteaux derivative of the extended cost functional (4.8) is

$$D\widetilde{J}(u,\tau)[v,\rho] = \mathbb{E}\Bigg[\int_0^\tau \big(D_x f(t,\tau,\theta_t)[Y_t^v] + D_u f(t,\tau,\theta_t)[v] + \rho D_\tau f(t,\tau,\theta_t)\big)\,\mathrm{d}t \\ + \rho f(\tau,\tau,\theta_\tau) + D_x h(X_\tau)^T Y_\tau^v + \mathcal{K}h(\tau,\theta_\tau)\rho\Bigg],$$

where

$$\mathcal{K}h(t,\theta_t) := D_x h(X_t^u)^T b(t,\theta_t) + \frac{1}{2}\mathrm{Tr}\left(\sigma(t,\theta_t)^T D_{xx} h(X_t^u)^T \sigma(t,\theta_t^u)\right),$$

is the Kolmogorov operator associated to the state SDE (4.1).

Proof. Since we have already assessed the Gâteaux differentiability of the running part of the cost functional in Section 4.1.1, we now only focus on the terminal time i.e. the cost functional is

$$\tilde{J}(u,\tau) = \mathbb{E}[h(X_\tau^u)].$$

Consider the perturbations of the admissible pair $(u,\tau) \in \mathcal{U}'_{ad} \times \mathcal{S}_T$:

$$u^\epsilon := u + \epsilon v, \qquad \tau^\epsilon := \tau + \epsilon\rho,$$

where $v \in \mathcal{U}$ and $\rho \in \mathcal{S}$, with $\epsilon > 0$ sufficiently small so that $\tau^\epsilon \in \mathcal{S}_T$ and $u^\epsilon \in \mathcal{U}'_{ad}$.

Then,

$$\begin{aligned}\frac{\widetilde{J}(u^\epsilon,\tau^\epsilon) - \widetilde{J}(u,\tau)}{\epsilon} &= \frac{1}{\epsilon}\mathbb{E}\Big[h(X_{\tau^\epsilon}^\epsilon) - h(X_\tau)\Big] \\ &= \underbrace{\frac{1}{\epsilon}\mathbb{E}\left[h(X_{\tau^\epsilon}^\epsilon) - h(X_\tau^\epsilon)\right]}_{I_1^\epsilon} + \underbrace{\frac{1}{\epsilon}\mathbb{E}\left[h(X_\tau^\epsilon) - h(X_\tau)\right]}_{I_2^\epsilon},\end{aligned}$$

where X^ϵ is the solution to the state equation (4.1) with control u^ϵ.

Now we can study the two terms separately as $\epsilon \to 0$. Considering I_1^ϵ, by Itô's formula 1.11 applied to $h(X^\epsilon)$ on $[\tau, \tau^\epsilon]$, we obtain

$$h(X_{\tau^\epsilon}^\epsilon) - h(X_\tau^\epsilon) = \int_\tau^{\tau^\epsilon} \mathcal{K}h(t, \theta_t^\epsilon)\mathrm{d}t + \int_\tau^{\tau^\epsilon} D_x h(X_t^\epsilon)\sigma(t, \theta_t^\epsilon)\mathrm{d}W_t,$$

Taking the expectations, the stochastic integral vanishes (being a martingale with zero mean since both σ and h have polynomial growth and $X \in L^p$). Hence, thanks to the continuity of $\mathcal{K}h$ in x

$$I_1^\epsilon = \frac{1}{\epsilon}\mathbb{E}\left[h(X_{\tau^\epsilon}^\epsilon) - h(X_\tau^\epsilon)\right] = \frac{1}{\epsilon}\mathbb{E}\left[\int_\tau^{\tau^\epsilon} \mathcal{K}h(t, \theta_t^\epsilon)\mathrm{d}t\right] \to \mathbb{E}\left[\mathcal{K}h(\tau, \theta_\tau^\epsilon)\rho\right].$$

For the second term, since h is differentiable and $X_\tau^\epsilon = X_\tau + \epsilon Y_\tau^v$, we have

$$\begin{aligned} I_2^\epsilon &= \frac{1}{\epsilon}\mathbb{E}[D_x h(X_\tau^\epsilon)^T(X_\tau^\epsilon - X_\tau)] + \frac{1}{\epsilon}O(|X_\tau^\epsilon - X_\tau|^2) \\ &= \mathbb{E}\left[D_x h(X_\tau)^T Y_\tau^v\right] + \underbrace{\frac{1}{\epsilon}O(|X_\tau^\epsilon - X_\tau|^2)}_{\to 0 \text{ as } \epsilon \to 0}. \end{aligned}$$

Combining these two results we get

$$\lim_{\epsilon \to 0} \frac{\widetilde{J}(u^\epsilon, \tau^\epsilon) - \widetilde{J}(u, \tau)}{\epsilon} = \mathbb{E}\left[D_x h(X_\tau)^T Y_\tau^v\right] + \mathbb{E}\left[\mathcal{K}h(\tau, \theta_\tau^\epsilon)\rho\right],$$

We can thus write the Gâteaux derivative of the cost functional with only the final contribution as

$$D\widetilde{J}(u, \tau)[v, h] = \mathbb{E}\left[D_x h(X_\tau) Y_\tau^v\right] + \mathbb{E}\left[\mathcal{K}h(\tau, \theta_\tau^\epsilon)\rho\right].$$

□

4.2.2 The Extended Adjoint Equation

We now introduce the extended adjoint equation in the same fashion as the process we did in the standard Pontryagin Maximum Principle in Section 3.4.

The *extended adjoint equation* is the following BSDE:

$$\begin{cases} \mathrm{d}p_t = -\Big(D_x f(t,\tau,\theta_t) + D_x b(t,\theta_t)^T p_t + \mathrm{Tr}(q_t^T D_x \sigma(t,\theta_t))\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_\tau = D_x h(X_\tau), \end{cases} \tag{4.9}$$

where X_t^u is the unique solution to the controlled state equation (4.1) associated to the control $u \in \mathcal{U}'_{ad}$.

We need to extend furthermore the solution of this BSDE to the interval $[0,T]$, by continuity $p_t \equiv D_x h(X_\tau)$ for all $t > \tau$. On the other hand both the terms needs to be zero so $q_t \equiv 0$ for all $t > \tau$ by the same reasoning as before, then the pair (p,q) solve the following BSDE until final time T.

$$\begin{cases} \mathrm{d}p_t = -\Big(\mathbb{1}_{t\le\tau} D_x f(t,\tau,\theta_t) + \mathbb{1}_{t\le\tau} D_x b(t,\theta_t)^T p_t + \mathrm{Tr}(q_t^T D_x \sigma(t,\theta_t))\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_T = D_x h(X_\tau). \end{cases} \tag{4.10}$$

To simplify the notation, we introduce the *extended Hamiltonian*:

$$\begin{aligned} \mathcal{H}(t,\tau,x,u,p,q) &: [0,T]^2 \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}, \\ \mathcal{H}(t,\tau,x,u,p,q) &:= \mathbb{1}_{\{t\le\tau\}} f(t,\tau,x,u) + \mathbb{1}_{\{t\le\tau\}} \langle b(t,x,u), p\rangle + \mathrm{Tr}(q^\top \sigma(t,x,u)) \\ D_x\mathcal{H}(t,\tau,x,u,p,q) &= \mathbb{1}_{\{t\le\tau\}} D_x f(t,\tau,x,u) + \mathbb{1}_{\{t\le\tau\}} \langle D_x b(t,x,u), p\rangle + \mathrm{Tr}(D_x\sigma(t,x,u) q^T). \end{aligned}$$

Note that this new version of the Hamiltonian is the same as the one introduced in the previous section on $[0,\tau]$, where there was not a final cost, since when $t > \tau$ we extended p_t to 0 canceling the term.

The adjoint equation (4.10) can then be rewritten, in terms of $\mathcal{H}$, as:

$$\begin{cases} \mathrm{d}p_t = -D_x\mathcal{H}(t,\tau,X_t^u,u_t,p_t,q_t)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_\tau = D_x h(X_\tau). \end{cases}$$

The pair (p_t, q_t) is to be intended as a solution to (4.10) in the sense of Definition 2.2.

Theorem 4.3 (Well-posedness of the adjoint equation)**.** Let assumptions ($\mathcal{P}$-1), ($\mathcal{P}'$-2) and ($\mathcal{P}'$-3) hold. Then, there exists a unique solution $(p,q) \in \mathbb{S}^2(\mathbb{R}^n) \times \mathbb{H}^2_\tau(\mathbb{R}^{n\times d})$ to the extended adjoint equation (4.10) in the sense of Definition 2.1.

Proof. The result follows from standard existence and uniqueness theorems for BSDEs once the assumptions of Assumptions ($\mathcal{H}$-2) are verified.

1. **Terminal condition.** $D_x h(X_T^u) \in L^2(\Omega;\mathbb{R}^n)$ which is true since $D_x h$ is $\mathcal{F}_T$-adapted by assumption and

$$D_x h(X_T^u) \leq C|X_T^u|^{q-1} \leq \infty,$$

since we know that $X \in L^p(\Omega\times[0,T];\,\mathbb{R}^n)$, $u \in L^p(\Omega\times[0,T];\,\mathbb{R}^k)$ and $p > 2q$.

2. **Integrability of the generator at zero.** The process $\{D_x\mathcal{H}(t,\theta_t,0,0), t \in [0,T]\}$ satisfies

$$D_x\mathcal{H}(t,\theta_t^u,0,0) = D_x f(t,\tau,\theta_t^u),$$

which is assumed to be have polynomial growth by assumption. Hence

$$\mathbb{E}\left[\int_0^T |D_x f(t,\tau,\theta_t^u)|\mathrm{d}t\right] \leq \mathbb{E}\left[\int_0^T C(1+|X_t^u|^q+|u_t|^q+|\tau|^q)\mathrm{d}t\right] < +\infty,$$

since $X_t \in L^p(\mathbb{R}^n)$ and $u_t \in L^p(\mathbb{R}^k)$ for some $p > 1$ and $\tau < T$ a.s.

3. **Lipschitz property.** For any $(p,q),(p',q') \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, by linearity of the inner product and of the trace, by the triangle inequality 1.3, there exists $L > 0$ such that for all $t \in [0,T]$,

$$\begin{aligned}|D_x\mathcal{H}(t,\theta_t^u,p,q)-&D_x\mathcal{H}(t,\theta_t,p',q')|\\ &= \left|\mathbb{1}_{\{t\leq\tau\}}(p-p')^\top D_x b(t,\theta_t) + \mathrm{Tr}\big((q-q')^\top D_x\sigma(t,\theta_t)\big)\right|\\ &\leq L\,(|p-p'|+|q-q'|)\end{aligned}$$

which shows that the generator is Lipschitz in (p,q).

Since all conditions of Assumptions ($\mathcal{H}$-2) are satisfied, by Theorem 2.1, there exists a unique adapted pair (p,q) solving (4.10). □

4.2.3 Representation of the Gradient of the Extended Cost Functional

Similarly to the result of Theorem 4.1, one can find that

$$\mathbb{E}\left[p_\tau^T Y_\tau^v\right] = \mathbb{E}\left[\int_0^\tau \left(\langle p_t^n, D_u b(t,\theta_t)[v]\rangle + \langle q_t^n, D_u\sigma(t,\theta_t)[v]\rangle - \mathbb{1}_{t<\tau} D_x f(t,\tau,\theta_t)^T Y_t^v\right)\mathrm{d}t\right].$$

Now note that, in this case

$$\mathbb{E}\left[p_\tau^T Y_\tau^v\right] = \mathbb{E}\left[D_x h(X_\tau)^T Y_\tau^v\right]$$

Theorem 4.4 (Representation of the Gradient)**.** Let Assumptions ($\mathcal{P}$-**1**), ($\mathcal{P}'$-**2**), ($\mathcal{P}'$-**3**) hold. Let $u \in \mathcal{U}'_{ad}$ and $v \in \mathcal{U}$, let $\tau \in \mathcal{S}_T$ and $\rho \in \mathcal{S}$, let X^u the solution to the state equation (4.1), Y^v the solution of the linearized equation (4.2) and let (p,q) the solution to the adjoint equation (4.10). Then, it holds

$$D\widetilde{J}(u,\tau)[v,\rho] = \mathbb{E}\Bigg[\int_0^T D_u\mathcal{H}(\theta_t,p_t,q_t)[v] + D_\tau\mathcal{H}(\theta_t,p_t,q_t)\cdot\rho\,\mathrm{d}t$$
$$+\frac{1}{2}\mathrm{Tr}\left(\sigma(\tau,\theta_\tau)^T h_{xx}(X_\tau)\sigma(\tau,\theta_\tau)\right)\rho\Bigg].$$

Proof. Recall the form of the gradient of the extended cost functional from Theorem 4.2

$$D\widetilde{J}(u,\tau)[v,\rho] = \mathbb{E}\Bigg[\int_0^T \mathbb{1}_{\{t\le\tau\}}\big(D_x f(t,\tau,\theta_t)[Y_t^v] + D_u f(t,\tau,\theta_t)[v] + \rho D_\tau f(t,\tau,\theta_t)\big)\,\mathrm{d}t$$
$$+\rho f(\tau,\tau,\theta_\tau) + D_x h(X_\tau)^T Y_\tau^v + \mathcal{K}h(\tau,\theta_\tau)\rho\Bigg],$$

From the terminal condition of the adjoint equation (4.10), we have

$$\mathbb{E}\left[p_\tau^T Y_\tau^v\right] = \mathbb{E}\left[D_x h(X_\tau)^T Y_\tau^v\right]$$

Observing that

$$\begin{aligned} D_\tau \mathcal{H}(t,\tau,X_t,u_t,p_t,q_t) &= \mathbb{1}_{\{t\le\tau\}} D_\tau f(t,\tau,X_t,u_t) + \delta(t-\tau) f(t,\tau,X_t,u_t) \\ &\quad + \delta(t-\tau)\langle b(t,X_t,u_t),p_t\rangle, \\ D_u \mathcal{H}(t,\tau,X_t,u_t,p_t,q_t) &= \mathbb{1}_{\{t\le\tau\}} D_u f(t,\tau,X_t,u_t) + \mathbb{1}_{\{t\le\tau\}}\langle D_u b(t,X_t,u_t),p_t\rangle \\ &\quad + \operatorname{Tr}(D_u\sigma(t,X_t,u_t)q_t^T), \end{aligned}$$

where we have intended the derivative of the indicator function as the Dirac delta. This was possible thanks to a similar argument as the one provided in the Section 4.1.1. Moreover we must note that the derivative with respect to τ in the points which are on the boundary of $t \le \tau$ is to be intended in the directions of the intern points in the set. In other words, the direction can be of both signs if $\tau = t < T$ and is negative when $t = \tau$. Then, the thesis follows immediately. □

The first term in the gradient represents the classical variation with respect to the control u, through the Hamiltonian gradient $D_u\mathcal{H}$. The second term represents the variation with respect to the stopping time τ. When $h \equiv 0$ and f does not depend on τ, this expression reduces to the classical gradient representation of the fixed-time stochastic problem.

Note that the term $\langle b(\theta_t), p_t\rangle$ in $\mathcal{K}h$ gets canceled out with the one in $D_\tau\mathcal{H}$, since both are equal when valued in $t = \tau$.

4.3 The Pontryagin Maximum Principle

We now conclude the analysis by deriving the first-order necessary conditions for optimality in the stochastic control problem with random terminal time and final cost. After having established the differentiability of the control-to-state map and obtained the adjoint representation of the gradient of the reduced cost functional, we are now in a position to characterize the structure of an optimal pair $(\overline{u}, \overline{\tau})$.

The following result provides a stochastic version of the *Pontryagin Maximum Principle* (PMP) adapted to our extended framework, where both the control process and the terminal time are optimization variables. These conditions generalize the classical Pontryagin Maximum Principle 3.6 to problems with random final time and terminal costs.

Theorem 4.5 (Pontryagin Maximum Principle)**.** Let Assumptions ($\mathcal{P}$-1), ($\mathcal{P}'$-2) and ($\mathcal{P}'$-3) hold, and let $\overline{u} \in \mathcal{U}_{ad}$ be an optimal control and $\overline{\tau} \in \mathcal{S}_T$ be an optimal stopping time for the problem. If $\overline{X} := X^{\overline{u}}$ is the solution to the state equation (4.1) and (p, q) the solution to the adjoint equation (4.10), then the following first-order necessary conditions hold.

(i) *Maximum condition for the control u.* For all $u \in \mathcal{U}_{ad}$,

$$\mathbb{E}\left[\int_0^{\overline{\tau}} D_u \mathcal{H}(t, \overline{\tau}, \overline{X}_t, \overline{u}_t, p_t, q_t)[u - \overline{u}]\mathrm{d}t\right] \geq 0.$$

Moreover if $\mathcal{H}$ is convex in u then, for every $(\mathcal{F}_t)$-stopping time $\rho \in [0, \overline{\tau}]$

$$\mathcal{H}(\rho, \overline{\tau}, \overline{X}_\rho, \overline{u}_\rho, p_\rho, q_\rho) \leq \mathcal{H}(\rho, \overline{\tau}, \overline{X}_\rho, u_\rho, p_\rho, q_\rho)$$

(ii) *Maximum condition for the control τ:* for all $\tau \in \mathcal{S}_T$, the following conditional stationarity holds almost surely:

$$\mathbb{E}\Bigg[\int_0^T D_\tau \mathcal{H}(t, \overline{\tau}, \overline{X}_t, \overline{u}_t, p_t, q_t)[\tau - \overline{\tau}]\,\mathrm{d}t + \frac{1}{2}\mathrm{Tr}\left(\sigma(\tau, \theta_\tau)^T D_{xx} h(X_\tau) \sigma(\tau, \theta_\tau)\right) \cdot (\tau - \overline{\tau})\Bigg] \geq 0.$$

Proof. The proof follows from the first variation formula derived in the previous sections and from the adjoint representation of the gradient.

Variation in the control u. This follows exactly the same Proof as in of Teorem 3.6.

Variation in the stopping time τ. Since $\mathcal{S}_T \subset L^2(\Omega; [0, T])$ is convex, for any alternative control $\tau \in \mathcal{S}_T$ and any $\epsilon \in (0, 1)$, the convex combination

$$\tau^\epsilon := (1 - \epsilon)\overline{\tau} + \epsilon\tau$$

also belongs to $\mathcal{S}_T$.

By the Gâteaux differentiability of the reduced cost functional 4.2.1,

$$\widetilde{J}(\overline{u}, \tau^{\epsilon}) = \widetilde{J}(\overline{u}, \overline{\tau}) + \epsilon \, D_{\tau}\widetilde{J}(\overline{u}, \overline{\tau})[\tau - \overline{\tau}] + o(\epsilon).$$

and since $\overline{\tau}$ is optimal, $\widetilde{J}(\overline{u}, \tau^{\epsilon}) - \widetilde{J}(\overline{u}, \overline{\tau}) \geq 0$ for all $\epsilon > 0$. Dividing by ϵ and letting $\epsilon \to 0^{+}$, we obtain

$$D_{\tau}\widetilde{J}(\overline{u}, \overline{\tau})[\tau - \overline{\tau}] \geq 0,$$

Using the full gradient representation derived in 4.4, the directional derivative in τ can be written as

$$\begin{aligned} D_{\tau}\widetilde{J}(\overline{u}, \overline{\tau})[\tau - \overline{\tau}] = \mathbb{E}\Bigg[\bigg(\int_0^T D_{\tau}\mathcal{H}\big(t, \overline{\tau}, \overline{X}_t, \overline{u}_t, p_t, q_t\big) \, \mathrm{d}t \\ + \frac{1}{2} \mathrm{Tr}\left(\sigma(\overline{\tau}, \overline{\theta}_{\overline{\tau}})^T D_{xx} h(\overline{X}_{\overline{\tau}}) \sigma(\overline{\tau}, \overline{\theta}_{\overline{\tau}}) \right) \bigg) [\tau - \overline{\tau}] \Bigg] \geq 0. \end{aligned}$$

□

5

Examples

5.1 Quadratic Stochastic Control Problem

A special case of optimal control problems are the one where the cost functionals are quadratic. This class of stochastic control problems are called *quadratic stochastic control* and constitute an important class of problems. Their mathematical structure makes them a perfect example to explain how the PMP works. In many cases, these problems have solutions which can be analyzed through Riccati-type differential equations.

We consider the quadratic stochastic control problem with dynamics:

$$\mathrm{d}X_t = b(t, X_t, u_t)\,\mathrm{d}t + \sigma(t, X_t, u_t)\,\mathrm{d}W_t, \qquad X_0 = x_0 \in \mathbb{R}^n,$$

such that the coefficients b, σ satisfy the assumptions ($\mathcal{P}$-1) and ($\mathcal{P}$-2) and the cost functional:

$$J(x, u) = \mathbb{E}\left[\frac{1}{2}\int_0^T \left(|X_s - \overline{X}|^2 + |u_s|^2\right)\mathrm{d}s + \frac{1}{2}|X_T - \overline{X}_T|^2\right],$$

where $\overline{X}$, $\overline{X}_T \in \mathbb{R}^n$ are given target values. The running and terminal costs are defined as

$$f(t, x, u) = \frac{1}{2}\left(|x - \overline{X}|^2 + |u|^2\right), \qquad h(x) = \frac{1}{2}|x - \overline{X}_T|^2.$$

These functions satisfy the assumptions ($\mathcal{P}$-2) are satisfied. The Hamiltonian of the problem is

$$\mathcal{H}(t,x,u,p,q) = \frac{1}{2}|x - \overline{X}_t|^2 + \frac{1}{2}|u|^2 + \langle b(t,x,u), p\rangle + \mathrm{Tr}\big(\sigma(t,x,u)^\top q\big).$$

The corresponding adjoint BSDE is

$$\begin{cases} \mathrm{d}p_t = -\Big((X_t - \overline{X}) + D_x b(t,X_t,u_t)^\top p_t + \sum_j D_x\sigma^j(t,X_t,u_t)^\top q_t^j\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_T = (X_T - \overline{X}_T). \end{cases}$$

The maximum condition of the stochastic Pontryagin's maximum principle 3.6 gives

$$\big\langle D_u\mathcal{H}(t,\overline{X}_t,\overline{u}_t,p_t,q_t), u - \overline{u}_t\big\rangle \geq 0, \quad \forall\, u \in \mathcal{U}_{\mathrm{ad}},\ \text{a.e. in } (t,\omega).$$

Since

$$D_u\mathcal{H}(t,x,u,p,q) = u + b_u(t,x,u)^\top p + \sum_j \sigma_u^j(t,x,u)^\top q^j =: u + g(u_t),$$

and if $\mathcal{H}$ is convex in u, the optimality condition becomes

$$\langle \overline{u}_t + g(u_t),\, u - \overline{u}_t\rangle \geq 0, \quad \forall\, u \in \mathcal{U}_{\mathrm{ad}}.$$

Hence, if $\mathcal{U}_{ad} \subseteq L^2(\Omega \times [0,T];\mathbb{R}^k)$ is closed, the optimal control is

$$\overline{u}_t = \mathrm{Proj}_{\mathcal{U}_{\mathrm{ad}}}(-g(u_t))\,,$$

where the projection is inteded in $L^2(\Omega \times [0,T];\mathbb{R}^k)$.

Note that $g(u_t)$ is dependent on $\overline{u}$, so this is formulation in implicit and needs to be solved. When b and σ are linear in the control, i.e.

$$\begin{aligned} b(t,x,u) &= u + \tilde{b}(x), \\ \sigma(t,x,u) &= u + \tilde{\sigma}(x) \end{aligned}$$

for some functions $\tilde{b}$ and $\tilde{\sigma}$. Then $g(u_t)$ is independent of the control u and so the projection would be explicit.

5.2 Extended Example with Terminal Control

Another interesting class of stochastic control problems are the one in which we are penalizing the stopping time. Indeed, in some situations we may want to discourage strategies that delay decision excessively. For example this setting is useful when timing an investment, but with a cost of waiting beyond informational benefits.

We consider the performance functional

$$J(x,u,\tau) = \mathbb{E}\left[\int_0^\tau \hat{f}(s,X_s,u_s)\,\mathrm{d}s + \frac{1}{p}\,\tau^p\right] = \mathbb{E}\left[\int_0^\tau \hat{f}(s,X_s,u_s) + s^{p-1}\,\mathrm{d}s\ \right],$$

with running cost

$$f(s,x,u) = \hat{f}(s,x,u) + s^{p-1}.$$

The extended Hamiltonian is given by

$$\mathcal{H}(t,\tau,x,u,p,q) = \mathbb{1}_{\{t\leq\tau\}} f(t,x,u) + \langle p, b(t,x,u)\rangle + \mathrm{Tr}(q^\top \sigma(t,x,u)).$$

For the control u, the first-order condition reads $\forall u \in \mathcal{U}_{\mathrm{ad}}$

$$\left\langle \hat{f}_u(\overline{\tau},\overline{X}_{\overline{\tau}},\overline{u}_{\overline{\tau}}) + b_u(\overline{\tau},\overline{X}_{\overline{\tau}},\overline{u}_{\overline{\tau}})^\top p_{\overline{\tau}} + \sum_j \sigma_u^j(\overline{\tau},\overline{X}_{\overline{\tau}},\overline{u}_{\overline{\tau}})^\top q_{\overline{\tau}}^j,\ u_{\overline{\tau}} - \overline{u}_{\overline{\tau}} \right\rangle \geq 0.$$

For the control τ, differentiating the cost term gives

$$\mathcal{H}_\tau(t,\tau,X,u,p,q) = \hat{f}(\tau,X_\tau,u_\tau) + \tau^{p-1}$$

leading to the variational inequality

$$\left\langle \hat{f}(\overline{\tau},\overline{X}_{\overline{\tau}},\overline{u}_{\overline{\tau}}) + \overline{\tau}^{p-1},\ \tau - \overline{\tau} \right\rangle \geq 0.$$

In the special case $p = 2$, we obtain the simplified condition

$$\left\langle \hat{f}(\overline{\tau},\overline{X}_{\overline{\tau}},\overline{u}_{\overline{\tau}}) + \overline{\tau},\ \tau - \overline{\tau} \right\rangle \geq 0, \quad \forall \tau \in \mathcal{S}_T.$$

Hence, the optimal terminal control is given by the implicit projection formula

$$\overline{\tau} = \mathrm{Proj}_{\mathcal{S}_T}\left(-\hat{f}(\overline{\tau}, \overline{X}_{\overline{\tau}}, \overline{u}_{\overline{\tau}})\right).$$

This formula can be solved with numerical methods. In particular, if $\hat{f}$ does not depend on t then this solution is explicit for $\overline{u}$.

Now let's incorporate the first example into this one, i.e.

$$J(x,u,\tau) = \mathbb{E}\Bigg[\frac{1}{2}\int_0^\tau \underbrace{\left(|X_s - \overline{X}|^2 + |u_s|^2\right)}_{\hat{f}} + 2s^{p-1}\,\mathrm{d}s + \underbrace{\frac{1}{2}|X_\tau - \overline{X}_\tau|^2}_{h}\Bigg]$$

In this new setting, the extended Hamiltonian is given by:

$$\begin{aligned}\mathcal{H}(t,\tau,x,u,p,q) = \mathbb{1}_{\{t\leq\tau\}}\frac{1}{2}\left(|x-\overline{X}|^2 + |u|^2 + 2s^{p-1}\right) \\ + \mathbb{1}_{\{t\leq\tau\}}\langle p, b(t,x,u)\rangle + \mathrm{Tr}(q^\top\sigma(t,x,u)),\end{aligned}$$

where (p,q) solve the follow BSDE with random terminal time:

$$\begin{cases}\mathrm{d}p_t = -\Big((X_t - \overline{X}) + D_x b(t,X_t,u_t)^\top p_t + \sum_j D_x\sigma^j(t,X_t,u_t)^\top q_t^j\Big)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \\ p_\tau = (X_\tau - \overline{X}_\tau).\end{cases}$$

Regarding the optimal control $u \in \mathcal{U}_{ad}$, the procedure is the same as the one in the first example, leading to the same result.

For the optimal terminal time, we have the variational inequality:

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(\frac{1}{2}(|\overline{X}_{\overline{\tau}} - \overline{X}|^2 + |\overline{u}_{\overline{\tau}}|^2 + 2\overline{\tau}^{p-1}) + \langle p_{\overline{\tau}}, b(\overline{\tau}, \overline{X}_{\overline{\tau}}, \overline{u}_{\overline{\tau}})\rangle \\ + \mathrm{Tr}\left(\sigma(\overline{\tau},\theta_{\overline{\tau}})^T\sigma(\overline{\tau},\theta_{\overline{\tau}})\right)\Bigg)\cdot(\tau - \overline{\tau})\Bigg] \geq 0.\end{aligned}$$

5.3 Valuation of natural resources

This example was inspired by Knudsen et al. (1998) where the problem is about evaluating an investment in industry under the assumptions that it produces a single commodity and its value depends on the commodity price as well as on the way in which production is scheduled. The controller chooses the production rate (which can be changed instantly and free of charge) and the production stop time. Moreover we make the assumption that the natural resource has an infinite amount of resources even if this hypothesis is unrealistic.

Following the notation used in Pham (2009), we consider the market price for the natural resource as the process X_t

$$dX_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t, \qquad X_0 = x_0 > 0.$$

where $\mu, \sigma > 0$ are constants.

The running profit of the firm is given by a nondecreasing function f of the price. The *real options value* of the firm is therefore given by the optimal stopping problem:

$$\widetilde{J}(\tau) = \mathbb{E}\left[\int_0^\tau e^{-\beta t} f(X_t)\mathrm{d}t\right],$$

where $\beta > 0$ denotes the discount rate. Although this formulation seeks to maximize this value, the results apply equally to minimization through the equivalence

$$\max f = -\min(-f).$$

We assume that f satisfies Assumptions ($\mathcal{P}$-**1**), ($\mathcal{P}$-**2**), ($\mathcal{P}$-**3**). Using the notation introduced in Section 4, the *extended Hamiltonian* is

$$\mathcal{H}(t, \tau, x, u, p, q) = \chi_{\{t \leq \tau\}} f(x) e^{-\beta t} + \mu\, x\, p + \sigma\, x\, \theta,$$

where the pair (p, q) satisfies the adjoint BSDE

$$\mathrm{d}p_t = -\mathcal{H}_x(t, \tau, X_t, u_t, p, q)\,\mathrm{d}t + q_t\,\mathrm{d}W_t, \qquad p_\tau = 0.$$

Since f is nondecreasing, three different cases arise:

1. $f(\infty) = \lim_{x\to\infty} f(x) \le 0$.
   If
   $$f(\infty) = \lim_{x\to\infty} f(x) \le 0,$$
   then $f(x) \le 0$ for all $x \ge 0$. The optimal strategy is thus to stop immediately, i.e. $\overline{\tau} = 0$. Indeed, the variational inequality of the extended PMP 4.5 is verified:
   $$\langle f(X_{\overline{\tau}})\, e^{-\beta\overline{\tau}}, \tau - \overline{\tau}\rangle \le 0,$$
   since
   $$\langle f(x_0), \tau\rangle \le 0, \qquad \forall \tau \in \mathcal{S}_T.$$
2. $f(\infty) \ge 0$ and $f(0) = \inf f(x) \ge 0$. It is optimal to continue producing as long as possible, i.e. $\overline{\tau} = T$. Indeed, since $\tau - T \le 0$ for all $\tau \in \mathcal{S}_T$,
   $$\langle f(X_T)e^{-\beta T}, \tau - T\rangle \le 0,$$
   and the variational inequality holds.
3. $f(0) < 0 < f(\infty)$. Here f crosses zero exactly once (or infinitely many times), and there exists a threshold x^* such that $f(x^*) = 0$. Define the stopping time
   $$\overline{\tau} := \inf\{t \ge 0 : X_t \le x^*\} \wedge T \in \mathcal{S}_T.$$
   Since X_t is continuous, we have $f(X_{\overline{\tau}}) = 0$ almost surely. Therefore, the variational inequality is trivially satisfied:
   $$\langle f(X_{\overline{\tau}})e^{-\beta\overline{\tau}}, \tau - \overline{\tau}\rangle = 0,$$

So our results confirm what was obtained with the Dynamic Programming approach in Pham (2009).

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